On the Local Langlands Correspondence for $\mathrm{GL}_2(F)$ When $p \neq 2$

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Abstract

This thesis is an exposition of the proof of the local Langlands correspondence for GL_2 over a nonarchimedean local field F whose residual characteristic is $p \neq 2$. The local Langlands correspondence may be seen as a sort of generalization of local class field theory. It posits the existence of a canonical bijection between the set of irreducible smooth representations of $\mathrm{GL}_2(F)$ and a special class of smooth representations, called *Weil-Deligne representations*, of the Weil group of F. Regarding irreducible smooth representations of $\mathrm{GL}_2(F)$, these are separated into two classes: those occuring in the *principal series* and *supercuspidal* representations. The first three chapters amount to providing a complete classification of these. In chapter 4, we understand the structure of Weil-Deligne representations and provide motivation for their consideration. The canonicity of the local Langlands correspondence manifests in the L-functions and epsilon factors associated to representations on either side agreeing. The theory of these L-functions is treated in chapter 5. The thesis concludes in chapter 6 with demonstrating the bijection between the aforementioned sets of representations and showing the desirable properties which it enjoys.

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Contents

Introduction

Let E/F be a finite extension of nonarchimedean local fields in a fixed separable closure F^{sep} of F. The results of local class field theory (cf. [SG13]) produce, via the cup product with a certain *fundamental class*, an isomorphism of Tate cohomology groups

$$\widehat{H}^{-2}(\mathrm{Gal}(E/F), \mathbb{Z}) \xrightarrow{\sim} \widehat{H}^{0}(\mathrm{Gal}(E/F), E^{\times}).$$

Upon unpacking the definitions of these cohomology groups, we get an isomorphism

$$\mathrm{Gal}(E/F)^{\text{ab}} \xrightarrow{\sim} F^{\times}/N_{E/F}(E^{\times}),$$

where $\mathrm{Gal}(E/F)^{\text{ab}}$ is the abelianization of the Galois group and $N_{E/F}$ is the field norm map. The inverse of this isomorphism is known as the *local reciprocity map* $\mathrm{art}_{E/F}$. Taking profinite completions on both sides then yields

$$\mathrm{art}_F \colon \varprojlim_{E/F} F^{\times}/N_{E/F}(E^{\times}) = \widehat{F^{\times}} \xrightarrow{\sim} \mathrm{Gal}(F^{\text{sep}}/F)^{\text{ab}}.$$

The multiplicative group $F^{\times}$ is a subgroup of its completion $\widehat{F^{\times}}$, and its image under art_F is the abelianization of the so-called *Weil group of* F, $\mathcal{W}_F$, which may be thought of as a modification of the absolute Galois group of F (a more legitimate treatment of $\mathcal{W}_F$ occurs in the fourth chapter).

Elementary representation theoretic considerations tell us that, as $F^{\times}$ is Abelian, all of its irreducible smooth representations (over $\mathbb{C}$, say) are characters of $F^{\times}$. That is, group homomorphisms

$$\chi \colon F^{\times} \to \mathbb{C}^{\times}.$$

On the other hand, any character of a group factors through its abelianization. Thus, the local reciprocity map art_F allows us to directly identify characters of $F^{\times}$ with those of $\mathcal{W}_F$ in a canonical way. Moreover, associated to multiplicative characters of local fields and to smooth representations of the Weil group of F one has *L-functions* and *local epsilon factors* (described in chapters 4 and 5). The fact $F^{\times}$ is isomorphic to $\mathcal{W}_F^{\text{ab}}$ means that these L-functions and local epsilon factors vacuously agree: a character of $F^{\times}$ is precisely a character of $\mathcal{W}_F^{\text{ab}}$.

The *local Langlands correspondence for* GL_n is a generalization of this phenomenon. The group $F^{\times}$ may also be regarded as $\mathrm{GL}_1(F)$ and a character of $\mathcal{W}_F$ we think of as a 1-dimensional smooth representation. The local Langlands correspondence for GL_n then demands a canonical bijection

$$\left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{smooth irreducible representations} \\ \text{of } \mathrm{GL}_n(F) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ n\text{-dimensional semisimple} \\ \text{Weil-Deligne representations} \end{array} \right\},$$

where a *Weil-Deligne representation* is a smooth representation of $\mathcal{W}_F$ with some added structure. The bijection is canonical in the sense that the L-functions and epsilon factors on both sides should agree under the correspondence.

The local Langlands correspondence for GL_n is nowadays a theorem, first proved by Laumon, Rapoport, and Stuhler in 1993 for F with positive characteristic ([LRS93]), and in 2001 by Harris and Taylor for F of characteristic 0 ([HT01]). This thesis is an exposition of the proof of the local Langlands correspondence for $\mathrm{GL}_2(F)$ when the residual characteristic of F is $p \neq 2$. Towards this end, we follow closely the excellent treatment of the topic given by Bushnell and Hennihart in their 2006 exposition ([BH06]).

As just mentioned, the results as described in this thesis are complete only when the residual characteristic of F is not 2. When this is the case, a certain class of irreducible smooth representations (called *supercuspidal*) are able to be exhaustively classified by using a technique involving tamely ramified quadratic extensions of F. While the correspondence holds for the case $p = 2$, its proof is considerably more complicated and involves relatively *ad hoc* methods. We say nothing of it here.

The first chapter of this thesis is dedicated to providing a sufficient background on the theory of smooth representations of locally profinite groups in order to proceed with demonstrating the local Langlands correspondence. A reader already well-versed in representation theory may very well skip it, referring back only as necessary. The remainder of the text may be split into three parts: The first consists of classifying all irreducible smooth representations of $\mathrm{GL}_2(F)$. This is the content of chapters 2 and 3. These representations are split into two classes depending on whether or not they contain the trivial character of a subgroup of unipotent matrices. Those which do turn out to all come from a process known as *parabolic induction*, and the classification of such representations follows with relative ease. Those which do not require much more work to get our hands on.

The second part concerns the representation theory of the Weil group and understanding Weil-Deligne representations. This is the content of the fourth chapter. Finally, the third part is "putting it all together," where we state and prove the local Langlands correspondence for $\mathrm{GL}_2(F)$ formally.

Chapter 1

Representation Theoretic Background

This first chapter is dedicated to providing a quick but essentially complete background on smooth representations of locally profinite groups which will be necessary for the content of this thesis. We also develop the representation theory of $\mathrm{GL}_2(k)$, for a finite field k. An understanding of the structure of locally profinite groups is assumed.

1.1 Smooth Representations of Locally Profinite Groups

Let G be a locally profinite group. In this section we introduce and study the properties of a special class of representations of G called smooth representations.

Definition 1.1.1. Let $\pi\colon G \to \mathrm{GL}(V)$ be a representation of G where V is a complex vector space. We say (π, V) (or sometimes just π, or just V) is *smooth* if for every $v \in V$, there exists a compact open subgroup $K \subseteq G$ such that $\pi(k)v = v$ for all $k \in K$.

We will use the notation V^K for the set of K-fixed vectors in V:

$$V^K = \{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}. \tag{1.1}$$

With this we can also characterize a representation (π, V) as being smooth if for all $v \in V$, there exists a compact open subgroup K of G such that $v \in V^K$. The representation (π, V) is said to be *admissible* if $\dim V^K < \infty$ for all compact open subgroups K of G.

Example 1. Let $G = \mathrm{GL}_2(F)$ for F a nonarchimedean local field and let X be the complex vector space of functions $f\colon G \to \mathbb{C}$ such that for each f there is a compact open K of G for which $f(gk) = f(g)$. Then we get a smooth representation of G, (π, X) defined by

$$\pi(g)f(x) = f(xg).$$

It is clear to see that this is smooth.

Our goal will be to classify and understand all smooth representations of the locally profinite group $\mathrm{GL}_2(F)$ for F a nonarchimedean local field. Some smooth representations admit decompositions into direct sums of other representations. Thus we want to give a notion of what the "building blocks" of representations are. In other words, the representations which cannot be decomposed any further.

Definition 1.1.2. A smooth representation (π, V) of G is *irreducible* if $V \neq 0$ and there are no nontrivial proper G-stable subspaces of V.

Suppose (π_1, V_1) and (π_2, V_2) are two smooth representations of G. Then a *map of representations* is a linear map $f\colon V_1 \to V_2$ such that

$$f \circ \pi_1(g) = \pi_2(g) \circ f \tag{1.2}$$

for all $g \in G$. In other words f is a G-equivariant map. We say the two smooth representations of G are *isomorphic* or *equivalent* if there exists a G-equivariant isomorphism of vector spaces between them.

Remark 1. The smooth representations of G as objects together with maps of representations as morphisms form an Abelian category which we will denote by $\mathrm{Rep}(G)$.

A *character* of G is a continuous group homomorphism $\chi\colon G \to \mathbb{C}^\times$. We note that any character of G is a smooth one-dimensional representation $(\chi, \mathbb{C})$. Conversely, any one-dimensional smooth representation of G is equivalent to a representation given by a character.

Remark 2. Some authors use the term *quasi-character* for the notion of character we have just defined, reserving the term character for those quasi-characters whose image lands in the unit circle $S^1 \subseteq \mathbb{C}^\times$. We will call such $\chi\colon G \to \mathbb{C}^\times$ with $\chi(G) \subseteq S^1$ *unitary characters*.

Definition 1.1.3. A smooth representation (π, V) of G is *semisimple* if it admits a decomposition

$$V = \bigoplus_{i \in I} V_i,$$

where each V_i is an irreducible G-subspace of V. If all smooth representations of G are semisimple then we say the category $\mathrm{Rep}(G)$ is *semisimple*.

The property of semisimplicity is a desirable one, as if the representations of G are semisimple then just understanding the irreducible representations of G is sufficient to understand all representations of G. In practice we are not so lucky and it turns out there exist many locally profinite groups G for which $\mathrm{Rep}(G)$ is not semisimple.

1.1.1 Smooth Induction and Frobenius Reciprocity

Suppose that (π, V) is a smooth representation of G, and $H \subseteq G$ is a subgroup. Then there is a very natural way to obtain a smooth representation of H, namely just restriction of π to H. This defines a functor (it is really just a forgetful functor)

$$\mathrm{Res}_H^G\colon \mathrm{Rep}(G) \to \mathrm{Rep}(H).$$

In what follows we will construct a functor in the opposite direction to restriction, called *smooth induction*. It will thus be a functor

$$\mathrm{Ind}_H^G\colon \mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

Let (σ, W) be a smooth representation of a subgroup $H \subseteq G$. We first will construct the underlying vector space of the induced representation $\mathrm{Ind}_H^G \sigma$. It is the complex vector space X of functions $f\colon G \to W$ such that

(i) $f(hg) = \sigma(h)f(g)$ for all $h \in H$, $g \in G$.

(ii) For each f there is a compact open subgroup $K \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G$, $k \in K$.

We then equip X with the structure of a smooth G representation by defining a homomorphism

$$\Sigma\colon G \to \mathrm{GL}(X)$$

via

$$(\Sigma(g)f)(x) = f(xg).$$

The fact that Σ is a homomorphism is clear by the definition, and the fact that (Σ, X) is a smooth representation follows from (ii) above. The smooth induced representation $\mathrm{Ind}_H^G \sigma$ is then defined to be the representation (Σ, X).

In some cases in our analysis of the representations of $\mathrm{GL}_2(F)$ we will see that the whole space of locally constant functions X is "too big." We introduce a modication of the induction functor called *compact induction*. The vector space is the same as that of an induced representation, with the additional condition that the functions $f\colon G \to W$ must be compactly supported modulo H. The action of G is still right translation. Compact induction is denoted by $\text{c-Ind}_H^G \sigma$, for σ a smooth representation of H.

The induction functor enjoys a property which will be used extensively throughout this exposition:

Theorem 1.1.4 (Frobenius Reciprocity)**.** *Let (σ, W) and (π, V) be smooth representations of H and G, respectively. Then there is a canonical isomorphism*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma),$$

functorial in both σ and π.

Proof. First we note that there is a canonical H-equivariant homomorphism

$$\mathrm{ev}_1\colon\ \mathrm{Ind}_H^G \sigma \to W$$

given by $f \mapsto f(1)$. Let $\varphi\colon V \to X$ be in $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma)$. The map

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \to \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma)$$

is then given by $\varphi \mapsto \mathrm{ev}_1 \circ \varphi$. Then given an H-equivariant homomorphism $f\colon V \to W$, we define a G-equivariant map $f_*\colon V \to X$ by

$$f_*\colon v \mapsto (g \mapsto f(\pi(g)v)).$$

The maps $\varphi \mapsto \mathrm{ev}_1 \circ \varphi$ and $f \mapsto f_*$ are mutually inverse. □

Frobenius reciprocity tells us that the functor Ind_H^G is the *right adjoint* to the functor Res_H^G.

A similar result is true for compact induction, with the caveat that the subgroup H must be open. Note that in this case, c-Ind_H^G is the *left adjoint* to Res_H^G.

Theorem 1.1.5. *Let H be an open subgroup of G, (σ, W) a smooth representation of H, and (π, V) a smooth representation of G. Then there is a canonical isomorphism*

$$\mathrm{Hom}_G(\text{c-Ind}_H^G \sigma, \pi) \cong \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \pi)$$

that is functorial in both σ and π.

1.1.2 Schur's Lemma and Duality

For the remainder of this section we will assume the hypothesis that for some compact open $K \subseteq G$, the set of cosets G/K is countable. Note that if this holds for one compact open then it holds for all compact opens. The hypothesis will hold for all groups which we are concerned with in the following chapters.

Lemma 1.1.6. *Let (π, V) be a smooth irreducible representation of G. Then $\dim V$ is at most countable.*

Proof. As π is smooth, for a vector $v \in V$, the set $\pi(G)v$ is in bijection with G/K for some compact open subgroup $K \subseteq G$. By our assumption, G/K is countable. If $v \neq 0$, then clearly $\mathrm{span}(\pi(G)v)$ is a nontrivial G-subspace of V, whence irreduciblity demands $\mathrm{span}(\pi(G)v) = V$. Therefore the dimension of V is at most countable. □

This lemma allows one to prove the following, more consequential lemma:

Lemma 1.1.7 (Schur's Lemma)**.** *If (π, V) is a smooth irreducible representation of G, then $\mathrm{End}_G(V) = \mathbb{C}$.*

Proof. Let $\varphi \in \mathrm{End}_G(V)$ be nonzero. Since the kernel and image of φ are both G-subspaces of V and (π, V) is assumed irreducible, then φ is necessarily invertible, whence $\mathrm{End}_G(V)$ is a division algebra over $\mathbb{C}$. Moreover, if we choose a nonzero $v \in V$, then its G-translates span V by irreduciblity, and so any $\varphi \in \mathrm{End}_G(V)$ is determined uniquely by the value of $\varphi(v)$. Thus we see, by the previous lemma, $\dim \mathrm{End}_G(V)$ is countable.

On the other hand, any $\varphi \in \mathrm{End}_G(V)$ such that $\varphi \notin \mathbb{C}$ generates a transcendental field extension $\mathbb{C}(\varphi) \subseteq \mathrm{End}_G(V)$. The subset $\{(\varphi - a)^{-1} \mid a \in \mathbb{C}\}$ is linearly independent in $\mathbb{C}(\varphi)$ over $\mathbb{C}$, implying $\mathbb{C}(\varphi)$ has uncountable dimension. This is a contradiction. □

Corollary 1.1.8. *If G is Abelian, then any smooth irreducible representation of G is one-dimensional.*

In particular, for any smooth irreducible representation (π, V) of G, when restricted to the center Z of G, π acts as a character. More formally there is a character $\omega_\pi \colon Z \to \mathbb{C}^\times$ such that

$$\pi(g)v = \omega_\pi(g)v$$

for all $g \in Z$. We call ω_π the *central character* of (π, V). If we are given a character χ, we say a representation (π, V) *admits χ as a central character* if $\omega_\pi = \chi$. The next proposition will play a critical role later in our exposition.

Proposition 1.1.9. *Let (π, V) be a smooth representation of G. Let K be an open subgroup of G such that KZ/Z is compact.*

(i) For $v \in V$, the KZ-span of v is of finite dimension and is a sum of irreducible KZ-spaces.

(ii) The restriction $\mathrm{Res}^G_{KZ}\, \pi$ is semisimple.

Proof. See [BH06, §2.7]. □

We now shift to discuss duality of smooth representations of locally profinite groups. Let (π, V) be such a representation. We denote the dual vector space of V by $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and the evaluation pairing by $\langle v^*, v\rangle$, $v^* \in V^*, v \in V$. We can equip V^* with the structure of a G-representation by defining $\pi^* \colon G \to \mathrm{GL}(V^*)$ with

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle.$$

Unfortunately this in general will not be smooth. To address this we define

$$\check{V} = (V^*)^\infty = \bigcup_K (V^*)^K$$

together with $\check{\pi} = (\pi^*)^\infty = \pi^*|_{(\check{V})}$. The resulting $(\check{\pi}, \check{V})$ is a smooth representation of G called the *contragredient* (or sometimes the *smooth dual*) of V. A key result regarding contragredient is the following:

Proposition 1.1.10. *Let (π, V) be an admissible representation of G. Then (π, V) is irreducible if and only if $(\check{\pi}, \check{V})$ is irreducible.*

Proof. See [BH06, §2.8]. □

1.2 Haar Integrals and the Hecke Algebra

We once again fix G a locally profinite group and henceforth denote by $C_c^\infty(G)$ the set of locally constant complex valued functions on G with compact support. The space $C_c^\infty(G)$ has two natural actions of G. Let $f \in C_c^\infty(G)$. Then for each $g \in G$ we have the *left tranlsation of f by g, $\lambda_g f$*:

$$(\lambda_g f)(x) = f(g^{-1}x),$$

and the *right translation of f by g, $\rho_g f$*:

$$(\rho_g f)(x) = f(xg).$$

The requirement that f be locally constant implies that both λ and ρ are smooth representations of G on $C_c^\infty(G)$.

1.2.1 Haar Integrals

For our purposes, the precise details of the proofs of existence and uniqueness of Haar integrals on a locally profintie group are less relevant than the properties they hold. Thus in this section we primarily state such results without proof, and direct the reader to [BH06, §3] or [RV98, §1.2] for full details.

Definition 1.2.1. A *right (resp. left) Haar integral* on G is a nonzero linear functional

$$I \colon C_c^\infty(G) \to \mathbb{C}$$

such that

(i) $I(\rho_g f) = I(f)$ (resp. $I(\lambda_g f) = I(f)$) for all $g \in G$, $f \in C_c^\infty(G)$.

(ii) $I(f) \geq 0$ for any $f \in C_c^\infty(G)$ such that $f \geq 0$.

The group G possesses a right Haar integral which is essentailly unqiue.

Proposition 1.2.2. *There exists a right Haar integral $I\colon C_c^\infty(G) \to \mathbb{C}$. Moreover, if I' is another Haar integral on G, then $I' = cI$ for some positve constant c.*

Corollary 1.2.3. *For $f \in C_c^\infty(G)$, let $\check{f} \in C_c^\infty(G)$ by $\check{f}(g) = f(g^{-1})$. The functional $I'\colon C_c^\infty(G) \to \mathbb{C}$ given by $I'(f) = I(\check{f})$ is a left Haar integral on G. Any left Haar integral on G is of the form cI' for some positive constant c.*

Let I be a left Haar integral and Ψ_S the characteristic function of a nonempty compact open subset S of G. We define

$$\mu_G(S) = I(\Psi_S).$$

By definition $\mu_G(S) > 0$, and also by definition we have $\mu_G(gS) = \mu_G(S)$ for any $g \in G$. We refer to μ_G as a left *Haar measure* on G. We henceforth use the notation

$$I(f) = \int_G f(g) d\mu_G(g).$$

When there is no possibility of confusion on the Haar measure being used, we will sometimes abbreviate the above as

$$\int_G f(g) dg.$$

Definition 1.2.4. The group G is *unimodular* if any left Haar integral on G is also a right Haar integral.

If μ_G is a left Haar measure on G, for any $g \in G$ we can define a functional $C_c^\infty(G) \to \mathbb{C}$ by

$$f \mapsto \int_G f(xg) d\mu_G(x).$$

It is easy to see that this functional is also a left Haar integral. Thus there is a positive real number (dependent on g) $\delta_G(g)$ such that

$$\delta_G(g) \int_G f(xg) d\mu_G(x) = \int_G f(x) d\mu_G(x)$$

for all $f \in C_c^\infty(G)$. This defines a homomorphism $\delta_G\colon G \to \mathbb{R}_{>0}$. Taking $f = \Psi_K$ for a compact open subgroup K, we find that δ_G is trivial on K and thus δ_G is in fact a character of G.

Definition 1.2.5. The character δ_G as defined above is called the *modular character* of G.

Remark 3. Observe that if $\delta_G = \mathbb{1}_G$ (the trivial character on G), then that amounts to saying the left Haar integral is invariant under right translation. In other words it is also a right Haar integral, whence G is unimodular. Conversely, if G is unimodular then the left Haar integral is right translation invariant and δ_G must be trivial.

Now let V be a complex vector space. We consider the space of locally constant functions $f\colon G \to V$ with compact support, which we denote by $C_c^\infty(G; V)$.

Proposition 1.2.6. *The space $C_c^\infty(G; V)$ is isomorphic to $C_c^\infty(G) \otimes V$.*

Proof. The isomorphism is given by sending $\sum_i f_i \otimes v_i \in C_c(G) \otimes V$ to the function $(g \mapsto \sum_i f_i(g) v_i) \in C_c^\infty(G; V)$. $\square$

For a left Haar measure μ_G, there is a unique linear functional $I_V : C_c^\infty(G; V) \to V$ such that

$$I_V(f \otimes v) = \left(\int_G f(g) d\mu_G(g) \right) \cdot v.$$

This defines a linear functional on all of $C_c^\infty(G; V)$ as the basic tensors $f \otimes v$ span the space $C_c^\infty(G) \otimes V \cong C_c^\infty(G; V)$. We use the notation, for $\varphi \in C_c^\infty(G; V)$,

$$I_V(\varphi) = \int_G \varphi(g) d\mu_G(g).$$

We introduce one more variation on the Haar integral. Let $H \subseteq G$ be a closed subgroup and let δ_H be the modular character of H. Fix a character θ of H. Denote by $C_c^\infty(H \backslash G, \theta)$ underlying vector space of the representation $\text{c-Ind}_H^G \theta$. Then we have:

Proposition 1.2.7. *Let $\theta \colon H \to \mathbb{C}^\times$ be a character of H. Then the following are equivalent:*

(i) There exists a nonzero linear functional $I_\theta \colon C_c^\infty(H \backslash G, \theta) \to \mathbb{C}$ such that $I_\theta(f) = I_\theta(\rho_g f)$ for all $g \in G$.

(ii) $\theta \delta_H = \delta_G|_H$.

In the case the assumptions of the propostion hold, the linear functional I_θ is like a Haar integral:

Corollary 1.2.8. *Suppose the conditions of proposition 1.2.7 are satisfied. Then there is a nonzero linear functinal I_θ on $C_c^\infty(H \backslash G, \theta)$ such that*

(i) $I_\theta(f) = I_\theta(\rho_g f)$ for all $g \in G$, and

(ii) $I_\theta(f) \geq 0$ for all $f \in C_c^\infty(H \backslash G, \theta)$ such that $f \geq 0$.

The linear functional I_θ is moreover uniquely determined up to a positive scalar factor.

Thus we use the notation

$$I_\theta(f) = \int_{H \backslash G} f(g) d\mu_{H \backslash G}(g)$$

for $f \in C_c^\infty(H \backslash G, \theta)$. The $\mu_{H \backslash G}$ is called a *positive semi-invariant measure on $H \backslash G$*. The main result of this discussion on Haar integrals is the following:

Theorem 1.2.9 (Duality Theorem)**.** *Let $H \subseteq G$ be a closed subgroup and $\dot{\mu}$ a positive semi-invariant measure on $H \backslash G$. Put*

$$\delta_{H \backslash G} = (\delta_H^{-1} \delta_G)|_H.$$

Suppose that (σ, W) is a smooth representation of H. Then there is a natural isomorphism

$$(\text{c-Ind}_H^G \sigma)^\vee \cong \text{Ind}_H^G \delta_{H \backslash G} \otimes \check{\sigma}$$

depending only on the choice of $\dot{\mu}$.

Proof. See [BH06, §3.5]. $\square$

1.2.2 The Hecke Algebra

For a finite group G, one naturally identifies a representation of G with a module over $\mathbb{C}[G]$. When G is a locally profinite group we can still consider a representation as a module over a certain algebra, namely the *Hecke algebra* $\mathcal{H}(G)$. In what follows we define $\mathcal{H}(G)$ and demonstrate that a smooth G-representation is an identical concept to a "smooth $\mathcal{H}(G)$-module."

The underlying vector space of $\mathcal{H}(G)$ is $C_c^\infty(G)$. We will assume that G is unimodular for this discussion, so as to not have to constantly be precise about left and right Haar integrals. Thus we fix a Haar measure μ on G and define a bilinear product $*$ on $C_c^\infty(G)$ as follows:

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x).$$

Bilinearity of $*$ is clear from the definition and associativity is an easy computation. Accordingly, we define $\mathcal{H}(G) = (C_c^\infty(G), *)$ to be the *Hecke Algebra* of G. It is an associative algebra over $\mathbb{C}$.

Remark 4. There are two facts about $\mathcal{H}(G)$ which we should carry with us:

(i) $\mathcal{H}(G)$ is unital if and only if G is finite.

(ii) $\mathcal{H}(G)$ is commutative when G is.

Let K be a compact open subgroup of G. We define $e_K \in \mathcal{H}(G)$ via

$$e_K(x) = \begin{cases} \mu(K)^{-1} & \text{if } x \in K \\ 0 & \text{if } x \notin K. \end{cases}$$

Proposition 1.2.10. *Let e_K be defined as above. Then:*

(i) The function e_K satisfies $e_K * e_K = e_K$.

(ii) A function $f \in \mathcal{H}(G)$ satisfies $e_K * f = f$ if and only if $f(kg) = f(g)$ for all $k \in K$, $g \in G$.

(iii) The space $e_K * \mathcal{H}(G) * e_K$ is a unital subalgebra of $\mathcal{H}(G)$ with identity e_K.

We will use the notation

$$\mathcal{H}(G, K) = e_K * H(G) * e_K.$$

Observe that $f \in \mathcal{H}(G, K)$ if and only if $f \in \mathcal{H}(G)$ and $f(k_1 g k_2) = f(g)$ for $k_1, k_2 \in K$. We also have

$$\mathcal{H}(G) = \bigcup_K \mathcal{H}(G, K), \tag{1.3}$$

with the union being taken over all comapct open subgroups of G.

For M a left module over $\mathcal{H}(G)$, we write $f * m$, $f \in \mathcal{H}(G)$, $m \in M$ for module action.

Definition 1.2.11. Let M be a left $\mathcal{H}(G)$-module. We say that M is a *smooth $\mathcal{H}(G)$-module* if $\mathcal{H}(G) * M = M$.

This terminology makes sense: By (1.3), we have that an $\mathcal{H}(G)$-module is smooth if and only if for every $m \in M$, there is some compact open subgroup K of G such that $e_K * m = m$.

Let (π, V) be a smooth representation of G. For $f \in \mathcal{H}(G)$, $v \in V$, we define $\Phi_{\pi,f} \in C_c^\infty(G; V)$ by

$$\Phi_{\pi,f}(g) = f(g)\pi(g)v.$$

We then define

$$\pi(f)v = \int_G \Phi_{\pi,f}(g)d\mu(g) \in V.$$

Proposition 1.2.12. *Let (π, V) be a smooth representation of G. The assignment*

$$(f, v) \mapsto \pi(f)v$$

gives V the structure of a smooth $\mathcal{H}(G)$-module. Moreover if (π', V') is another smooth G-representation and $\varphi\colon V \to V'$ is a G-homomorphism, then φ is also a $\mathcal{H}(G)$-homomorphism.

We can also go the other way:

Proposition 1.2.13. *Let M be a smooth $\mathcal{H}(G)$-module. Then there is a unique G-homomorphism*

$$\pi\colon G \to \mathrm{GL}(M)$$

such that (π, M) is a smooth G-representation and for $f \in \mathcal{H}(G)$, $m \in M$,

$$\pi(f)m = f * m.$$

Moreover, if M' is a smooth $\mathcal{H}(G)$-module with associated smooth representation (π', V'), then any $\mathcal{H}(G)$-homomorphism $M \to M'$ is a G-homomorphism $V \to V'$.

For proofs of both propositions, see [BH06, §4.2]. With these we see that smooth representations of G and smooth $\mathcal{H}(G)$-modules are interchangable concepts.

1.3 The Representation Theory of $\mathrm{GL}_2(k)$

In this section we give a classification of the irreducible representations of $\mathrm{GL}_2(k)$ for k a finite field of q elements. Our motivation for this is two-fold: Firstly, on a macro-level the techniques used in studying the representation theory of $\mathrm{GL}_2(k)$ provide a sort of roadmap for how we will approach the more formidable task of doing the same for $\mathrm{GL}_2(F)$, F nonarchimedean. Secondly, in the study of the representations of $\mathrm{GL}_2(F)$, where k is the residue field of F, we will sometimes inflate representations of $\mathrm{GL}_2(k)$ to those $\mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is the integer ring of F. Then, we induce to a representation in $\mathrm{GL}_2(F)$. Thus understanding representations of $\mathrm{GL}_2(k)$ will be essential for understanding those of $\mathrm{GL}_2(F)$.

1.3.1 The Structure of $\mathrm{GL}_2(k)$

For the remainder of this section let $G = \mathrm{GL}_2(k)$. There are some important subgroups of G which we consider (the analogues of these will appear again when we study $\mathrm{GL}_2(F)$). The first of which is the *standard Borel subgroup*

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G \right\}$$

of upper triangular matrices in G. We also have

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G \right\},$$

which is the *standard maximal split torus* in G (we will often just call it the torus), and the *unipotent radical*

$$N = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in G \right\}.$$

Observe that there is an extension

$$1 \to N \to B \to T \to 1,$$

whence we have the semi-direct product decomposition $B = T \ltimes N$. Finally we put Z for the center of G.

Using row-reduction operations, one shows that for G we have the *Bruhat decomposition*

$$G = B \cup BwB,$$

where

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a permutation matrix. Thus we have $\{1, w\}$ as coset representatives for the double coset space $B \backslash G / B$. The following is a result of elementary linear algebra techniques and counting arguments:

Lemma 1.3.1. *The group G has order $(q^2-1)(q^2-q)$ and has q^2-1 conjugacy classes.*

Thus it follows that G possesses q^2-1 irreducible representations up to equivalence.

1.3.2 Principal Series Representations

Our first method of constructing irreducible representations begins with fixing characters χ_1, χ_2 of $k^\times$. We then form the character χ of T defined by

$$\chi = \chi_1 \otimes \chi_2 \colon \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \chi_1(a)\chi_2(b). \tag{1.4}$$

Since $T \cong B/N$, we may consider a character of T as a character of B which is trivial on N. We thus can form the induced representation $\mathrm{Ind}_B^G \chi$ of G.

Lemma 1.3.2. *Let (π, V) be an irreducible representation of G. The following are equivalent:*

(i) π is isomorphic to a G-subspace of $\mathrm{Ind}_B^G \chi$ for some character χ of T, considered as a character of B trivial on N, and

(ii) π contains the trivial character of N, in the sense that $\pi|_N$ contains the trivial character of N as a subrepresentation.

Proof. First we remark that π containing a trivial character of N is equivalent to π containing an irreducible B-representation σ containing the trivial character of N. Then this is in turn equivalent to σ being the inflation of a character χ of T, since a representation of B trivial on N naturally defines a representation of T, but T is Abelian and so that representation is one-dimensional.

Now we apply Frobenius reciprocity:

$$\mathrm{Hom}_B(\mathrm{Res}_B^G \pi, \sigma) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi),$$

and the result follows. □

For a character χ of T defined as in (1.4), we introduce the associated character

$$\chi^w\colon \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \chi_2(a)\chi_1(b).$$

We think of χ^w as a character of B trivial on N as well.

Proposition 1.3.3. *Let χ, ξ be characters of T, which we view as characters of B trivial on N. Then:*

(i) The space $\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi)$ is trivial unless $\chi = \xi$ or $\chi = \xi^w$.

(ii) The spaces $\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \chi)$ and $\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \chi^w)$ have the same dimension. This dimension is 2 if $\chi = \chi^w$, or 1 otherwise.

Proof. We begin with Frobenius reciprocity:

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi) \cong \mathrm{Hom}_B(\mathrm{Res}_B^G(\mathrm{Ind}_B^G \chi), \xi). \tag{1.5}$$

Then, applying Mackey's restriction formula, we obtain

$$\mathrm{Res}_B^G(\mathrm{Ind}_B^G \chi) = \sum_{y \in B\backslash G/B} \mathrm{Ind}_{B\cap B^y}^B(\mathrm{Res}_{B\cap B^y}^B \chi^y), \tag{1.6}$$

where by χ^y we understand the character $b \mapsto \chi(yby^{-1})$ of the group $B^y = y^{-1}By$. The Bruhat decomposition of G tells us that in our case we only need to consider $y = 1$ and $y = w$. For $y = 1$, the corresponding summand on the right side of (1.6) is

$$\mathrm{Ind}_B^B(\mathrm{Res}_B^B \chi) = \chi.$$

For $y = w$, observe that

$$B^w = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in G \right\},$$

whence $B \cap B^w = T$. Thus the corresponding summand is

$$\mathrm{Ind}_T^B \chi^w.$$

From this, we can rewrite the right side of (1.5) as

$$\mathrm{Hom}_T(\chi, \xi) \oplus \mathrm{Hom}_B(\mathrm{Ind}_T^B \chi^w, \xi).$$

We apply Frobenius reciprocity once more for

$$\mathrm{Hom}_B(\mathrm{Ind}_T^B \chi^w, \xi) \cong \mathrm{Hom}_T(\chi^w, \xi).$$

Thus

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi) \cong \mathrm{Hom}_T(\chi, \xi) \oplus \mathrm{Hom}_T(\chi^w, \xi).$$

Then (i) follows immediately, and (ii) follows from taking $\xi = \chi$. □

Corollary 1.3.4. *Let χ be a character of T regarded as that of B which is trivial on N. Then the representation $\mathrm{Ind}_B^G \chi$ is irreducible if and only if $\chi \neq \chi^w$. If $\chi = \chi^w$, then $\mathrm{Ind}_B^G \chi$ has length 2 with distinct composition factors.*

The representations which are obtained by inducing a character of T to G (and their irreducible components) are said to be in the *principal series*. In other words the principal series consists of all the irreducible representations of G which contain the trivial character of N. One can count to find that G has $\frac{1}{2}(q^2+q)-1$ irreducible representations in the principal series.

1.3.3 Cuspidal Representations

We say an irreducible representation of G not containing the trivial character of N is a *cuspidal representation*. The cuspidal representations do not arise directly from induction, and as such their construction and classification proves to be a more laborious task compared to that of the principal series. The same will be true of $\mathrm{GL}_2(F)$ for F nonarchimedean.

In order to construct the cuspidal representations, we first begin with a quadratic field extension l/k, which is uniquely determined up to k-isomorphism. A character θ of the multiplicative group $l^\times$ is said to be *regular* if $\theta^q \neq \theta$. We identify l with the k-vector space $k \oplus k$ by choosing a basis, whence G is identified with $\mathrm{GL}(l)$. Then, the action of $l^\times$ on l identifies $l^\times$ with a subgroup E of G. Let θ be a regular character of E and fix ψ a nontrivial character of N. We define a character $\theta_\psi \colon ZN \to \mathbb{C}^\times$ by

$$\theta_\psi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} u\right) = \theta(a)\psi(u),$$

for $a \in k^\times, u \in N$.

Remark 5. The definition of θ_ψ is independent of ψ up to equivalence, as all nontrivial characters of N are conjugate.

Recall for a group G we can form the *representation ring of G, $R(G)$*. The elements of $R(G)$ are formal differences of isomorphism classes of finite dimensional representations of G, with addition being given by direct sum and multiplication by tensor product. An element of $R(G)$ is called a *virtual representation*. The main result is the following:

Theorem 1.3.5. *Let θ be a regular character of E and ψ a nontrivial character of N.*

(i) The virtual representation

$$\pi_\theta = \mathrm{Ind}_{ZN}^G \theta_\psi - \mathrm{Ind}_E^G \theta$$

is an irreducible cuspidal representation of G, with $\dim \pi_\theta = q-1$.

(ii) Let θ_1, θ_2 be regular characters of E. Then $\pi_{\theta_1} \cong \pi_{\theta_2}$ if and only if $\theta_1 = \theta_2$ or $\theta_1 = \theta_2^q$.

(iii) Every irreducible cuspidal representation of G is of the form π_θ for some regular character θ of E.

Proof. A counting argument shows that if (i) and (ii) hold then (iii) follows. Thus we show (i) and (ii) directly. We do this by computing the characters of π_θ, and find:

$$\begin{aligned}\operatorname{tr}\pi_\theta(z) &= (q-1)\theta(z), \;\; z \in Z, \\ \operatorname{tr}\pi_\theta(zu) &= -\theta(z), \;\; z \in Z, u \in N, u \neq 1, \\ \operatorname{tr}\pi_\theta(y) &= -(\theta(y) + \theta^q(y)), \;\; y \in E \smallsetminus Z.\end{aligned}$$

The character of π_θ is just 0 on all g not conjugate to some element of $E \cup ZN$, thus the above is the complete character table. The character table gives (ii) immediately. To see that π_θ is irreducible, we compute

$$\frac{1}{|G|}\sum_{g\in G} |\operatorname{tr}\pi_\theta(g)|^2 = 1.$$

Finally,

$$\sum_{u\in N} \operatorname{tr}\pi_\theta(u) = 0,$$

and so $\pi_\theta|_N$ does not contain the trivial character of N. Therefore π_θ is cuspidal. □

Chapter 2

The Principal Series for $\mathrm{GL}_2(F)$

For this chapter we denote by F a nonarchimedean local field. Unless otherwise stated we put $G = \mathrm{GL}_2(F)$. We fix the following notation which will be carried throughout the rest of this thesis:

$$\begin{aligned}
\mathcal{O} &= \text{the ring of integers in } F. \\
\mathfrak{p} &= \text{the maximal ideal in } \mathcal{O}. \\
\varpi &= \text{a prime element in F} \\
\mathfrak{k} &= \text{the residue field } \mathcal{O}/\mathfrak{p}. \\
q &= |\mathfrak{k}| \\
v_F(\cdot) &= \text{the discrete normalized valuation on } F. \\
\|\cdot\| &= \text{the absolute value on } F\ (\|x\| = q^{-v_F(x)}).
\end{aligned}$$

When there is possible confusion about the field F to which each of the above are associated, each will be decorated with a subscript indicating the field to which they belong ($\mathcal{O}_F$, for example).

In §1.3 we saw that the irreducible representations of $\mathrm{GL}_2(k)$, for k a finite field, can be classified as either in the principal series (i.e., the representation is equivalent to a $\mathrm{GL}_2(k)$-subspace of the induction of a character on the torus) or cuspidal. The group $\mathrm{GL}_2(F)$ exists in a similar paradigm, and we again distinguish representations by those in the principal series and those which are *supercuspidal*. The precise definitions of these two classes of representations have differences from those for $\mathrm{GL}_2(k)$, but the philosophy will be familiar.

2.1 The Structure of $\mathrm{GL}_2(F)$

We begin by remarking that $G = \mathrm{GL}_2(F)$ has direct analogues of the subgroups B, N, and T as were defined in §1.3. They are defined just the same and are all closed subgroups of G. We once again have isomorphisms $B/N \cong T$ and $B \cong T \ltimes N$, both of which are algebraic and topological.

The Bruhat decomposition (§1.3.1) holds for G as well. Changing the base field to F yields decompositions which were not present in the case of $\mathrm{GL}_2(k)$.

Proposition 2.1.1 (Iwasawa Decomposition). *Let $K = \mathrm{GL}_2(\mathcal{O})$. Then $G = BK$.*

Proof. Let $g \in G$. If the (2,1)-entry of g is zero, then $g \in B$ and we are done. Otherwise, we can multiply g on the right by a permutation matrix (which lies in K) to ensure that

$v_F(g_{(2,1)}) \geq v_F(g_{(2,2)})$. Finally we may multiply by a lower triangluar matrix in K on the right to make $g_{(2,1)} = 0$. □

An immediate consequence of the Iwasawa decomposition is that the right coset space $B \backslash G$ is a continuous image of $\mathrm{GL}_2(\mathcal{O})$, which we know to be compact. Therefore $B \backslash G$ is compact. We also have:

Proposition 2.1.2 (Cartan Decomposition)**.** *The matrices*

$$\begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix}$$

$a, b \in \mathbb{Z}$ such that $a \leq b$ are a set of representatives for the double coset space $K \backslash G / K$.

Proof. See [BH06, §7.2] □

It is a corollary of the Cartan decomposition that the set G/K is countable for any compact open subgroup $K \subseteq G$, whence G satisfies the assumption put in §1.1.2. The group G has another significant subgroup: The *standard Iwahori subgroup* is

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in \mathcal{O}^\times, c \in \mathfrak{p}, b \in \mathcal{O} \right\}.$$

For our immediate needs, the Iwahori subgroup will not play a crucial role, but later it shall be pivotal.

2.1.1 Haar Integrals on G and its Subgroups

Let $A = \mathrm{Mat}_2(F)$ be the ring of 2×2 matrices with coefficients in F. Then the additive group of A is isomorphic to $F \times F \times F \times F$. We form a Haar measure μ on A by choosing a Haar measure ν on F and setting

$$\mu = \otimes^4 \nu.$$

Proposition 2.1.3. *The group G is unimodular.*

Proof. Let μ be a Haar measure on A as defined above. For $\Phi \in C_c^\infty(G)$ we define $f_\Phi \in C_c^\infty(A)$ by

$$f_\Phi(x) = \begin{cases} 0 & \text{if } x \in A \setminus G \\ \Phi(x) \| \det x \|^{-2} & \text{if } x \in G. \end{cases}$$

Then the linear functional

$$\Phi \mapsto \int_A f_\Phi(x) d\mu(x)$$

is both a left and right Haar integral on G. □

Regarding the subgroups B, N, and T, there is really nothing to be said about N and T, as $N \cong F$ and $T \cong F^\times \times F^\times$. The Borel subgroup B, on the other hand, is not unimodular and hence admits a nontrivial modular character δ_B (as in §1.2.1). We recall that we have the semi-direct product decomposition $B \cong T \ltimes N$.

Proposition 2.1.4. *The modular character δ_B of B is given by*

$$\delta_B(tn) = \|t_1/t_2\|$$

for

$$t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}.$$

Proof. See [BH06, §7.6] □

Corollary 2.1.5. *Let δ_B be defined as in the proposition. The space $C_c^\infty(B\backslash G, \delta_B^{-1})$ admits a positive semi-invariant Haar measure $\dot{\mu}$. There is a Haar measure μ on $\mathrm{GL}_2(\mathcal{O})$ such that*

$$\int_{B\backslash G} f(g)d\dot{\mu}(g) = \int_{\mathrm{GL}_2(\mathcal{O})} f(x)d\mu(x)$$

for $f \in C_c^\infty(B\backslash G, \delta_B^{-1})$.

Recall that the Iwasawa decomposition yielded that $B\backslash G$ is compact. Thus any function on G is vacuously compactly supported modulo B. This means if σ is a smooth representation on T, viewed as a smooth representation of B as usual, we have

$$\text{c-}\mathrm{Ind}_B^G \sigma \cong \mathrm{Ind}_B^G.$$

We now apply theorem 1.2.9 to see that there is a canonical isomorphism

$$(\mathrm{Ind}_B^G \sigma)^\vee \cong \mathrm{Ind}_B^G \delta_B^{-1} \otimes \check{\sigma},$$

depending only on a choice of postive semi-invariant measure $\dot{\mu}$ on $C_c^\infty(B\backslash G, \delta_B^{-1})$.

2.2 Jacquet Modules and the Irreducibility Criterion

Let (π, V) be a smooth representation of G. We denote by V_N the N-coinvariants of V. That is,

$$V_N = V/V(N),$$

where $V(N) = \mathrm{span}\{v - \pi(n)v \mid n \in N\}$. The space V_N is then the largest quotient of V on which N acts trivially. Since, as we just stated, π restricted to N acts trivially on V_N, it carries a representation π_N of $B/N \cong T$. The pair (π_N, V_N) is called the *Jacquet module* of V (at N, but we shall only be concerned with this case). The *Jacquet functor* is the functor

$$\mathrm{Rep}(G) \to \mathrm{Rep}(T) \tag{2.1}$$
$$(\pi, V) \mapsto (\pi_N, V_N). \tag{2.2}$$

Once again we consider a smooth representation of B which is trivial on N to be a smooth representation of T. Suppose σ is such a representation. Then if (π, V) is a smooth G-representation we have

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \sigma) \cong \mathrm{Hom}_B(\mathrm{Res}_B^G \pi, \sigma). \tag{2.3}$$

Our discussion in the previous paragraph yields that (2.3) can be written

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \sigma) \cong \mathrm{Hom}_T(\pi_N, \sigma).$$

Our motivation for introducing the Jacquet module can be found in the following:

Proposition 2.2.1. *Let (π, V) be a irreducible smooth representation of G. Then the Jacquet module $V_N \neq 0$ if and only if π is isomorphic to a G-subspace of $\mathrm{Ind}_B^G \chi$ for some character χ of T.*

Proof. First suppose that π is isomorphic to some G-subspace of $\mathrm{Ind}_B^G \chi$. Then

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) \neq 0.$$

But Frobenius reciprocity gives $\mathrm{Hom}_T(\pi_N, \chi) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi)$, whence $V_N \neq 0$. Conversely, we observe that V_N admits an irreducible T-quotient of V_N. Since T is Abelian an irreducible representation is necessarily a character χ. Then $\mathrm{Hom}_T(\pi_N, \chi)$ is nontrivial which yields the desired result.

For full details of the construction of the T-quotient see [BH06, §9.1] □

An irreducible representation of G for which the Jacquet module is nontrivial is said to be a part of the *principal series*, whereas those with trivial Jacquet module are said to be *supercuspidal*. The remainder of this chapter is dedicated to providing a full classification of the former type of representation.

It is a fact that every irreducible smooth representation of G is admissible. The proof for the case of supercuspidal representations requires techniques not yet developed, and we will address it in the next chapter. We can deal with the other case now.

Proposition 2.2.2. *Let (π, V) be an irreducible smooth representation of G which is not cuspidal. Then (π, V) is admissible.*

Proof. Since any π which is not supercuspidal is a G-subspace of $\mathrm{Ind}_B^G \chi$ for some character χ of T, it suffices to show the claim for $\mathrm{Ind}_B^G \chi = (\Sigma, X)$. Let $K \subseteq \mathrm{GL}_2(\mathcal{O})$ be a compact open subgroup of G. Then

$$X^K = \{f \in X \mid f(bgk) = \chi(b)f(g) \text{ for } b \in B, g \in G, k \in K\}.$$

Since $G = B\,\mathrm{GL}_2(\mathcal{O})$ (cf. proposition 2.1.1), the set of double cosets $B\backslash G/K$ is finite. Each such double coset supports at most a one-dimensional space of functions in X^K (cf. [BH06, §3.5]). □

Lemma 2.2.3. *Let (σ, U) be a smooth T-representation and let (Σ, X) be the induction $\mathrm{Ind}_B^G \sigma$. Then there is an exact sequence of T-representations*

$$0 \to \sigma^w \otimes \delta_B^{-1} \to \Sigma_N \to \sigma \to 0,$$

where σ^w is the representation

$$t \mapsto \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

and the map $\Sigma_N \to \sigma$ is given by $f \mapsto f(1)$.

Proof. See [BH06, §9.3] □

Suppose that φ is a character of $F^\times$ and (π, V) a smooth representation of G. Then we form *the twist of π by φ*, denoted by $(\varphi\pi, V)$, by setting $\varphi\pi(g) = \varphi(\det g)\pi(g)$. It is clear to see that $\varphi\pi$ is also a smooth representation of G. A similar construction is carried out for twisting characters of T. Suppose $\chi = \chi_1 \otimes \chi_2$ is a character of T and φ as before. Then we put

$$\varphi \cdot \chi = \varphi\chi_1 \otimes \varphi\chi_2.$$

The main result of the section which will enable us to provide a classification of those representations of $\mathrm{GL}_2(F)$ which lie in the principal series is the following:

Theorem 2.2.4 (Irreducibility Criterion)**.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and $(\Sigma, X) = \operatorname{Ind}_B^G \chi$.*

(i) (Σ, X) is reducible if and only if $\chi_1\chi_2^{-1}$ is either the trivial character on $F^\times$ or the character $x \mapsto \|x\|^2$ on $F^\times$.

(ii) If (Σ, X) is reducible, then:

(a) X has G-composition length 2.

(b) One composition factor of X is of dimension one, while the other is infinite dimensional.

(c) X has a one-dimensional G-subspace if and only if $\chi_1\chi_2^{-1} = 1$.

(d) X has a one-dimensional G-quotient if and only if $\chi_1\chi_2^{-1} \colon x \mapsto \|x\|^2$.

The remainder of this section is dedicated proving the irreducibility criterion and understanding its consequences. In what follows we are considering χ and $(\Sigma, X) = \operatorname{Ind}_B^G \chi$ as in the theorem statement. First we define the B-subspace of X

$$V = \{f \in X \mid f(1) = 0\}.$$

We see then that $V_N = \ker(\Sigma_N \xrightarrow{\alpha_\chi} \chi)$ as was defined in lemma 2.2.3. The aforementioned lemma then gives that $V_N \cong \chi^w \otimes \delta_B^{-1}$. We put $V(N)$ for the kernel of the projection $V \to V_N$.

Proposition 2.2.5. *The kernel $V(N)$ of the projection $V \to V_N$ (notation of the preceding paragraph) is irreducible over B*

Proof. See [BH06, §9.7] □

The proposition gives the B-composition of X

$$V(N) \subseteq V \subseteq X$$

of length 3. The composition factors $V/V(N) = V_N \cong \chi^w \otimes \delta_B^{-1}$ and $X/V \cong \mathbb{C}$ are both of dimension one. In particular this means that the G-composition length of X is at most three.

We will need the following lemma in our proof of the irreducibility criterion. As usual, we put $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Lemma 2.2.6. *Let $f \in X$. Then $f \in V$ if and only if there exists a compact open subgroup $N_0 \subseteq N$ (which depends on f) such that* $\operatorname{supp} f \subseteq BwN_0$.

Proof. The condition for f being in V is that $f(1) = 0$. By definition the functions $f \in X$ are G-smooth, and so if $f(1) = 0$ then f also vanishes on some compact neighborhood of 1 which we can write BN_0', where N_0' is a compact open subgroup of

$$N' = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in G \right\}.$$

Then one has the identity

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} w \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

when $x \neq 0$. The right hand side of the above lies in $Bw\left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)$. The result then follows. □

We use this to prove:

Proposition 2.2.7. *For $X = \operatorname{Ind}_B^G \chi$ where $\chi = \chi_1 \otimes \chi_2$, the following are equivalent:*

(i) $\chi_1 = \chi_2$.

(ii) *X has a one-dimensional N-subspace.*

Suppose that these conditions hold. Then the one-dimensional N-subspace is unique, call it X_0, and the space X_0 is also a G-subspace which is not contained in V.

Proof. Observe that for a character φ of $F^\times$, one has

$$\operatorname{Ind}_B^G(\varphi \cdot \chi) = \operatorname{Ind}_B^G(\varphi\chi_1 \otimes \varphi\chi_2) = \varphi \operatorname{Ind}_B^G \chi.$$

Thus if (i) holds, we can take $\chi_1 = \chi_2 = 1$, and twist by a character of $F^\times$ which does not materially affect the structure of $\operatorname{Ind}_B^G \chi$. In this case, let $f\colon G \to \mathbb{C}$ be any nonzero constant function. The span of f is a one-dimensional G-stable subspace of X. Thus (i) implies (ii).

In the converse direction, let $f \in X$ span a dimension one N-subspace. The N-action on f is by right translation:

$$n \cdot f(x) = f(xn).$$

The space $\mathbb{C}f$ is N-stable and so $n \cdot f = \psi(n)f$ for some character ψ of N. We have the decomposition $G = B \cup BwN$. Now as $f \in X = \operatorname{Ind}_B^G \chi$, left translation by B does not affect the support of f. Since right translation by N is just multiplying by a character, the support of f is thus either all of G or BwN. In the latter case, $f(1) = 0$ and hence we are in the situation of lemma 2.2.6. But that means $\operatorname{supp} f \subseteq BwN_0$ for some compact open subgroup of N. Thus the second case is impossible and $\operatorname{supp} f = G$. Hence f vanishes nowhere and in particular $f(1) \neq 0$, so $\mathbb{C}f \not\subseteq V$.

The N-projection $X \to X/V \cong \mathbb{C}$ identifies $\mathbb{C}f$ with $\mathbb{C}$ as N-spaces ($\mathbb{C}$ being considered as a trivial N-space), and so it follows that N fixes f under right translation. Let $x \in F^\times$. We have the identity

$$w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

If $\|x\|$ is sufficiently large, then $\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)$ becomes close to the identity matrix, so the G-smoothness of f means that right translation by $\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)$ fixes f. Thus for $\|x\|$ large we have

$$f\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix}\right).$$

The far right of the above is of the form $f(nt)$ for $n \in N$ and $t \in T$. Thus, since $f \in X$, this becomes

$$f\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi_1(-1)\chi_1^{-1}\chi_2(x)f(1).$$

But f is invariant under right translation by N, so

$$f(w) = \chi_1(-1)\chi_1^{-1}\chi_2(x)f(1).$$

Surely the evaluation of f at w should not depend on x. Thus we have $\chi_1 = \chi_2$. Therefore (ii) implies (i). If we put $\varphi = \chi_1 = \chi_2$, then we have $f(g) = \varphi(\det g)f(1)$, which shows that the space $\mathbb{C}f$ is stable under G-action. The construction shows the space is uniquely determined, and we remarked before that it is not contained in V. □

We now are equipped to prove theorem 2.2.4.

Proof of the Irreducibility Criterion. Suppose that X is reducible. Then we determined previously that its G-composition length is either 2 or 3, and it has either a finite dimensional G-subspace or a finite dimensional G-quotient. Let us first consider the case that X has a finite dimensional G-subspace.

In this case, X has a one-dimensional N-subspace and so we are in the situation of proposition 2.2.7. Thus X has a one-dimensional G-subspace, X_0, and $\chi = \chi_1 \otimes \chi_2$ where $\chi_1 = \chi_2 = \varphi$, say. The group G acts on X_0 via $\varphi \circ \det$ and $X_0 \cap V = 0$, all as in the proposition. The quotient $Y = X/X_0$ is then isomorphic to V as B-representations. Since $\dim X_0 = 1$, if X has G-length 3 then Y has G-length 2. Recall, though, that $Y \cong V$ (B-isomorphic) has composition length 2, since V has an irreducible subrepresentation of B given by $V(N)$. In particular, V has a unique one-dimensional B-quotient, which gives a G-quotient of Y. This yields a contradiction, however, as it would force there to be a nontrivial factor in the Jacquet module Y_N, which is already one dimensional by lemma 2.2.3. Thus X must have G-composition length 2, and we see we are in case (ii)(c) of theorem 2.2.4.

In the case that X has a finite dimensional G-quotient, then the contragredient $\check{X}$ has a finite dimensional G-subspace, and we are in the situation of the previous case. Then the fact that $\check{X} \cong \mathrm{Ind}_B^G \delta_B^{-1} \otimes \check{\chi}$ along with proposition 2.1.4 gives that we are in case (ii)(d) of theorem 2.2.4.

The converse is handled by proposition 2.2.7 for when $\chi_1\chi_2^{-1} = 1$ and making a similar argument using duality for when $\chi_1\chi_2^{-1}(x) = \|x\|^2$. □

2.3 Classification of the Noncuspidal Representations

We conclude this chapter by arriving at a complete classification of the irreducible smooth representations of G for which the Jacquet module is not trivial.

Proposition 2.3.1. *Let χ and ξ be characters of T. Then*

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi) = \begin{cases} 1 & \text{if } \xi = \chi \text{ or } \xi = \chi^w \otimes \delta_B^{-1}. \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Frobenius reciprocity gives

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi) = \mathrm{Hom}_T((\mathrm{Ind}_B^G \chi)_N, \xi).$$

Then $(\mathrm{Ind}_B^G \chi)_N$ fits into the exact sequence (lemma 2.2.3)

$$0 \to \chi^w \otimes \delta_B^{-1} \to (\mathrm{Ind}_B^G \chi)_N \to \chi \to 0.$$

In the case $\chi \neq \chi^w \otimes \delta_B^{-1}$, then the above splits and the result follows. In the case $\chi = \chi^w \otimes \delta_B^{-1}$, we observe that for $x \in F^\times$ this means

$$\chi_1(x) = \|x\|\chi_2(x).$$

Then by theorem 2.2.4, $\mathrm{Ind}_B^G \chi$ is irreducible, whence the result. □

Suppose that χ is a character of T and $\mathrm{Ind}_B^G \chi$ is reducible. Then by the irreducibility criterion, there is a character φ of $F^\times$ such that $\chi = \varphi \cdot \mathbb{1}_T$ or $\chi = \varphi \cdot \delta_B^{-1}$. For now assume that $\varphi = 1$. Then we get an exact sequence

$$0 \to \mathbb{1}_G \to \mathrm{Ind}_B^G \mathbb{1}_T \to \mathrm{St}_G \to 0, \tag{2.4}$$

where $\mathrm{Ind}_B^G \mathbb{1}_T$ has an irreducible G-quotient St_G called the *Steinberg representation*. The situation is similar when $\varphi \neq 1$. Let $\varphi_G = \varphi \circ \det$ and $\varphi_T = \varphi \cdot \mathbb{1}_T = \varphi \otimes \varphi$. Then we get

$$0 \to \varphi_G \to \mathrm{Ind}_B^G \varphi_T \to \varphi \cdot \mathrm{St}_G \to 0.$$

Representations of the form $\varphi \cdot \mathrm{St}_G$ for character φ of $F^\times$ are called *special representations*.

To handle the case of $\chi = \varphi \cdot \delta_B^{-1}$, again suppose first that $\varphi = 1$. Then observe that taking the contragredient of the exact sequence (2.4) yields

$$0 \to (\mathrm{St}_G)^\vee \to \mathrm{Ind}_B^G \delta_B^{-1} \to \mathbb{1}_G \to 0.$$

Then proposition 2.3.1 implies $(\mathrm{St}_G)^\vee \cong \mathrm{St}_G$, so there is really nothing more to say. Before stating the final classification theorem, we introduce a slight modification on the induction functor to allow for more elegant statements.

Definition 2.3.2. If σ is a smooth representation of T, we define the *normalized induction* functor $\boldsymbol{\iota}_B^G(-) \colon \mathrm{Rep}(T) \to \mathrm{Rep}(G)$ by

$$\boldsymbol{\iota}_B^G \sigma = \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma).$$

With this, it is just unpacking definitions to see we can restate theorem 2.2.4 as:

Proposition 2.3.3. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T. Then $\boldsymbol{\iota}_B^G \chi$ is reducible if and only if $\chi = \varphi \cdot \delta_B^{\pm 1/2}$ for some character φ of $F^\times$.*

Collecting our work in this chapter, we arrive at the main result.

Theorem 2.3.4 (Classification of Noncuspidal Representations)**.** *The following is a list of all irreducible noncuspidal smooth representations of G up to isomorphism:*

(i) Induced representations $\boldsymbol{\iota}_B^G \chi$ for χ a character of T such that $\chi \neq \varphi \cdot \delta_B^{-1/2}$ for any character φ of $F^\times$.

(ii) One-dimensional representations $\varphi \circ \det$ where φ is a character of $F^\times$.

(iii) Special representations $\varphi \cdot \mathrm{St}_G$ where φ is a character of $F^\times$.

Chapter 3

Supercuspidal Representations of $\mathrm{GL}_2(F)$

Let (π, V) be an irreducible smooth representation of G. In the last chapter we determined (proposition 2.2.1) that when the Jacquet module, (π_N, V_N), of (π, V) is nontrivial then π is obtained by the induction of a character of the torus in G. Proposition 2.2.1 is an if and only if, and so to get our hands on supercuspidal representations (those irreducible smooth representations of G with trivial Jacquet module) we will need to employ new techniques.

The overall strategy involves considering filtrations of the groups $K = \mathrm{GL}_2(\mathcal{O})$ and I (the standard Iwahori subgroup), and examining the largest subgroups in these filtrations in which an irreducible smooth representation π has a fixed vector. It turns out that the representation occuring in π of the next largest subgroup in the filtration carries much information and will lead us to our classification of supercuspidal representations. There is much subtlety in the process.

We begin this chapter with tying up a loose end and showing that supercuspidal representations are admissible, whence all irreducible smooth representations of G are admissible. Afterwards we have the construction of supercuspidal representations as described in the previous paragraph.

3.1 Admissibility of Supercuspidals

It what follows representations of G are always assumed to be smooth unless otherwise stated.

We begin by introducing an alternative notion of a representation being supercuspidal. If (π, V) is a representation of G, then for each $v \in V$ and $\check{v} \in \check{V}$ we get a smooth function on G

$$\gamma_{v\otimes\check{v}} \colon G \to \mathbb{C}$$

defined by

$$g \mapsto \langle \check{v}, \pi(g)v \rangle.$$

We define $\mathcal{C}(\pi)$ to be the space spanned by all the functions $\gamma_{v\otimes\check{v}}$. The functions in $\mathcal{C}(\pi)$ are called *matrix coefficients* of π. If $\gamma_{v\otimes\check{v}}$ is the matrix coefficient associated to $(v, \check{v}) \in V \otimes \check{V}$ and π is irreducible, then for $z \in Z$ the center of G, one has

$$\begin{aligned}\gamma_{v\otimes\check{v}}(zg) &= \langle \check{v}, \pi(zg)v \rangle \\ &= \langle \check{v}, \omega_\pi(z)\pi(g)v \rangle \\ &= \omega_\pi(z)\gamma_{v\otimes\check{v}}(g),\end{aligned}$$

where ω_π is the central character of π. It follows that for any $\gamma \in \mathcal{C}(\pi)$ we have

$$\gamma(zg) = \omega_\pi(z)\gamma(g).$$

Therefore the support of $\gamma \in \mathcal{C}(\pi)$ is invariant under Z-translation.

Definition 3.1.1. An irreducible smooth representation (π, V) of G is said to be γ-*cuspidal* if every $\gamma \in \mathcal{C}(\pi)$ is compactly supported modulo Z.

The key fact, which is not at all obvious, is that a representation (π, V) is supercuspidal in the sense of having trivial Jacquet module if and only if it is γ-cuspidal. We will not prove this here, but instead direct the reader to [BH06, §10.2].

Proposition 3.1.2. *If (π, V) is an irreducible γ-cupsidal representation, then π is admissible.*

Proof. Suppose by way of contradiction that π is not admissible. Let $K \subseteq G$ be a compact open subgroup such that $\dim V^K$ is infinite. This dimension is countable, which implies that the dimension of $\check{V}^K \cong \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ is uncountable.

For a fixed nonzero $v \in V$, we define a map $\Gamma_v \colon \check{V}^K \to \mathcal{C}(\pi)$ by $\check{v} \mapsto \gamma_{v \otimes \check{v}}$. This map must be injective, as the irreducibility of π means the set $\{\pi(g)v \mid g \in G\}$ spans V, so $\Gamma_v(\check{v})$ is 0 if and only if $\check{v}$ is 0. Observe that the image of Γ_v is the space of functions $f \colon G \to \mathbb{C}$ such that $f(zkgk') = \omega_\pi(z)f(g)$ for all $z \in Z$, $k, k' \in K$, and $g \in G$ and such that f is supported on a finite union of cosets $ZKgK$. This implies that $\dim \Gamma_v(\check{V}^K)$ is at most countable, but Γ_v is an injective map into a space of uncountable dimension, so we arrive at a contradiction. □

This proposition combined with proposition 2.2.2 and the fact that every supercuspidal representation is γ-cuspidal gives the desirable result.

Corollary 3.1.3. *Every irreducible smooth representation of G is admissible.*

We also will use the following in constructing supercuspidal representations, which greatly reduces the amount of work needed to check γ-cuspidality.

Proposition 3.1.4. *Let (π, V) be an irreducible admissible representation of G. If some nonzero matrix coefficient of π is compactly supported modulo Z then π is γ-cuspidal.*

Proof. See [BH06, §10.1] □

3.2 Intertwining

Our first method of constructing supercuspidal representations involves the compact induction of representations of open subgroups of G which are compact modulo the center Z of G. When those representations satisfy certain *intertwining* properties, then the resulting representation of G will turn out to be supercuspidal. Of course, before any of that, we must understand what is meant by "intertwining." Denote by $\widehat{K}$ the set of isomorphism classes of irreducible smooth representations of a compact open subgroup K of G.

Definition 3.2.1. Let K_1 and K_2 be compact open subgroups of G and $\rho_1, \rho_2 \in \widehat{K}_1, \widehat{K}_2$, respectively. An element $g \in G$ *intertwines* ρ_1 *with* ρ_2 if

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0,$$

where ρ_1^g is the representation of $K_1^g = g^{-1}K_1 g$ given by $x \mapsto \rho_1(gxg^{-1})$.

Observe that the property of intertwining ρ_1 with ρ_2 as above depends only on the double coset $K_1 g K_2$. That is, g intertwines ρ_1 with ρ_2 only if all elements of $K_1 g K_2$ intertwine ρ_1 with ρ_2.

Definition 3.2.2. Let K be a compact open subgroup of G and $\rho \in \widehat{K}$. A smooth representation (π, V) of G *contains* ρ (or ρ *occurs in* π) if

$$\mathrm{Hom}_K(\rho, \pi) \neq 0.$$

Proposition 3.2.3. *Let K_1, K_2 be compact open subgroups of G and $\rho_1, \rho_2 \in \widehat{K}_1, \widehat{K}_2$, respectively. If (π, V) is an irreducible smooth representation of G in which both ρ_1 and ρ_2 occur (in the sense of definition 3.2.2), then there exists $g \in G$ such that g intertwines ρ_1 with ρ_2.*

Proof. The hypothesis that (π, V) contains ρ_1 and ρ_2 is equivalent to the isotypic space V^{ρ_i} being nonzero for $i = 1, 2$. Let $e_2 \colon V \to V^{\rho_2}$ be the K-projection onto the ρ_2-isotypic space. Since π is irreducible, the sets $\pi(g^{-1})V^{\rho_1} = V^{\rho_1^g}$, $g \in G$, span V. Thus there is some $g \in G$ such that $e_2 \circ \pi(g^{-1})$ induces a nonzero map $V^{\rho_1} \to V^{\rho_2}$. This g is an element which intertwines ρ_1 with ρ_2. □

We say two representations ρ_1, ρ_2 *intertwine* in G if there exists $g \in G$ intertwining ρ_1 with ρ_2. Similarly, an element $g \in G$ *intertwines* a representation ρ if g intertwines ρ with itself. The main result concerning intertwining is the following:

Theorem 3.2.4. *Let K be an open subgroup of G which contains and is compact modulo Z. Let (ρ, W) be an irreducible smooth representation of K. If an element $g \in G$ intertwines ρ if and only if $g \in K$ then the compactly induced representation $\text{c-Ind}_K^G \rho$ is irreducible and supercuspidal.*

Proof. Put $(\Sigma, X) = \text{c-Ind}_K^G \rho$. We define a K-embedding $\varphi^0 \colon W \to X$ by $\varphi^0(w) = \varphi^0_w$, where φ^0_w has support K and $(\varphi)^0_w(k) = \rho(k)w$. Then $\varphi^0(W)$ is the space of functions of X with support contained in K (cf. [BH06, §2.5 Lemma]) and we identify W with this space.

Since K and G are unimodular, the Duality Theorem (1.2.9) implies that $\check{X} \cong \mathrm{Ind}_K^G \check{\rho}$, which contains $\text{c-Ind}_K^G \check{\rho}$ as a G-subspace. The canonical K-embedding of $\check{W}$ in its compact induction $\text{c-Ind}_K^G \check{\rho}$ then identifies $\check{W}$ iwth the space of functions in $\check{X}$ with support in K. We now take nonzero $\check{w} \in \check{W} \subseteq \check{X}$ and nonzero $w \in W \subseteq X$. The matrix coefficient $\gamma_{w \otimes \check{w}}$ is nonzero and has support in K. Thus we just need to show $\text{c-Ind}_K^G \rho$ is irreducible and supercuspidality will follow by proposition 3.1.4 and the fact that γ-cuspidalilty is an equivalent notion to supercuspidality.

The compact induction X breaks up into a direct sum of its K-isotypic components, and thus any K-map $W \to X$ has image lying in X^ρ. Therefore we have

$$\mathrm{Hom}_K(W, X^\rho) = \mathrm{Hom}_K(W, X).$$

By Frobenius reciprocity for compact inductions, we have

$$\mathrm{Hom}_K(W, X) \cong \mathrm{End}_G(X).$$

We define the *ρ-spherical Hecke algebra of* G, $\mathcal{H}(G,\rho)$, to be the space of functions $f\colon G \to \mathrm{End}_G(W)$ which are compactly supported modulo Z and such that

$$f(kgk') = \rho(k)f(g)\rho(k'),$$

for $g \in G$, $k, k' \in K$. Let $\dot{\mu}$ be a Haar measure on G/Z. Then, for $\varphi_1, \varphi_2 \in \mathcal{H}(G,\rho)$ we define the convolution product

$$\varphi_1 * \varphi_2(g) = \int_{G/Z} \varphi_1(x)\varphi_2(x^{-1}g)d\dot{\mu}(x).$$

With this, $\mathcal{H}(G,\rho)$ is an associative unital algebra over $\mathbb{C}$. We now define a map $\mathcal{H}(G,\rho) \to \mathrm{End}_G(X)$ by

$$\varphi \mapsto \left((g \mapsto f(g)) \mapsto \left(g \mapsto \int_{G/Z} \varphi(x)f(x^{-1}g)d\dot{\mu}(x)\right)\right).$$

This map is an isomorphism of $\mathbb{C}$-algebras (cf. [BH06, §11.3 Lemma]). The hypothesis on intertwining implies that the dimension of $\mathcal{H}(G,\rho)$ is 1, whence the dimension of $\mathrm{Hom}_K(W, X^\rho)$ is 1. We thus conclude that $W = X^\rho$.

Let Y be a nonzero G-subspace of X. Then $0 \neq \mathrm{Hom}_G(Y,X) \subseteq \mathrm{Hom}_G(Y, \mathrm{Ind}_K^G \rho) \cong \mathrm{Hom}_K(Y,\rho)$, with the isomorphism holding by Frobenius reciprocity. By proposition 1.1.9, Y is semisimple over K, and so $Y^\rho \neq 0$. Since W is irreducible over K, we have $W = X^\rho \subseteq Y$. Because W generates X over G, we conclude that $Y = X$, and thus X is irreducible.

□

3.3 First Construction

We now begin with the construction of supercuspidal representations. The procedure that follows is more concrete than would allow for us to immediately gain a characterization of supercuspidal representations, but is indicative of the strategies we will employ in subsequent sections.

Let $K = \mathrm{GL}_2(\mathcal{O})$ and $K_1 = 1 + \mathfrak{p}\,\mathrm{Mat}_2(\mathcal{O})$. Then $K/K_1 \cong \mathrm{GL}_2(\mathfrak{k})$. We also define the group $I_1 = 1 + \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$. Observe that I_1 is the preimage in K of $N(\mathfrak{k})$ (the upper-triangular unipotent matrices in $\mathrm{GL}_2(\mathfrak{k})$). With this notation, we have:

Theorem 3.3.1. *Let (π, V) be an irreducible smooth representation of G that contains the trivial character of K_1. Then one and only one of the following holds:*

(i) π contains a representation λ of K which is inflated from an irreducible cuspidal representation $\tilde{\lambda}$ of $\mathrm{GL}_2(\mathfrak{k})$.

(ii) π contains the trivial character of I_1.

In the case that (i) holds, then π is supercuspidal and there exists a representation Λ of ZK such that $\Lambda|_K \cong \lambda$ and

$$\pi \cong \text{c-}\mathrm{Ind}_{ZK}^G \Lambda.$$

Before we can prove theorem 3.3.1 we need a lemma.

Lemma 3.3.2. *Let $\tilde{\rho}_1, \tilde{\rho}_2$ be irreducible representations of $\mathrm{GL}_2(\mathfrak{k})$ and let ρ_1, ρ_2 denote their respective inflations to representations of K. Suppose that $\tilde{\rho}_1$ is cuspidal. then*

(i) The representations ρ_1 and ρ_2 intertwine in G if and only if $\tilde{\rho}_1 \cong \tilde{\rho}_2$.

(ii) An element $g \in G$ intertwines ρ_1 if and only if $g \in ZK$.

Proof. See [BH06, §11.5] □

With this we may prove the theorem.

Proof of theorem 3.3.1. The group K stabilizes V^{K_1}. Thus, we can take the isotypic decomposition of V^{K_1} and consider it as a direct sum of irreducible representations of K, all of which are trivial on K_1. In other words, each is inflated from a representation of $\mathrm{GL}_2(\mathfrak{k})$. Let λ be such a representation inflated from $\tilde{\lambda}$. Then if $\tilde{\lambda}$ is not cuspidal, then $\tilde{\lambda}$ contains the trivial character of $N(\mathfrak{k})$, which means λ contains the trivial character of I_1 and we are in case (ii) of the theorem. Otherwise $\tilde{\lambda}$ is cuspidal and we are in case (i). The lemma and proposition 3.2.3 show that these cannot occur simultaneously.

Suppose now that $\tilde{\lambda}$ is cuspidal. Then π contains a representation Λ of ZK extending λ. Thus we have a nontrivial ZK-homomorphism $\Lambda \to \mathrm{Res}^G_{ZK}\,\pi$. By Frobenius reciprocity, this gives a nontrivial G-homomorphism $\text{c-Ind}^G_{ZK}\,\Lambda \to \pi$. The second part of the lemma and theorem 3.2.4 imply that $\text{c-Ind}^G_{ZK}\,\Lambda$ is irreducible and supercuspidal. Thus $\pi \cong \text{c-Ind}^G_{ZK}\,\Lambda$ and π is supercuspidal. □

3.4 Chain Orders and Strata

In the last section we saw that if an irreducible representation has some K_1-fixed vector, then the action of K on the space V^{K_1} can show if the representation is supercuspidal. In general, one needs to look "deeper" in the groups K and I to understand the behavior of a representation. For this we develop *chain orders* and *strata*. A chain order is the Lie algebra of a special type of subgroup of GL_2 which stabilizes chains of $\mathcal{O}$-lattices in $V = F \oplus F$ called a *parahoric subgroup*. The parahoric subgroups admit natural filtrations to which we associate characters of the n-th filtration group which is trivial on the $(n+1)$-th filtration group. An irreducible representation (π, V) of G then may contain such a character. This behavior, similar to the theorems of the previous section, will give insight whether or not π is supercuspidal. A useful isomorphism relates the chain orders to the group of characters of the filtration groups of the parahoric subgroups. Thus we are able to take advantage of the structure of the chain orders (which also admit natural filtrations) in our analysis.

3.4.1 Lattice Chains and Associated Orders

Let $V = F \oplus F$ so that we identify $G = \mathrm{GL}_2(F) \cong \mathrm{GL}(V)$. We put $A = \mathrm{Mat}_2(F) \cong \mathrm{End}_F(V)$.

Definition 3.4.1. An *$\mathcal{O}$-lattice chain* in V is a nonempty set

$$\mathcal{L} = \{L_i \mid i \in \mathbb{Z}\}$$

where each L_i is an $\mathcal{O}$-lattice in V such that $L_i \supsetneq L_{i+1}$ and $\mathcal{L}$ is stable under translation by $F^\times$.

Observe that, since $\mathcal{L}$ is stable under $F^\times$-translation, there is a positive integer $e_{\mathcal{L}}$ such that

$$xL_i = L_{i+v_F(x)e_{\mathcal{L}}},$$

for all $x \in F^\times$ and $L_i \in \mathcal{L}$. Up to transformation by some $g \in G$, the structure of lattice chains in V is quite controlled.

Proposition 3.4.2. *Let $\mathcal{L}$ be a lattice chain in V. Then $e_{\mathcal{L}} = 1$ or $e_{\mathcal{L}} = 2$ Moreover, if $e_{\mathcal{L}} = 1$, there exists $g \in G$ such that*

$$gL_i = \mathfrak{p}^i \oplus \mathfrak{p}^i$$

for all $L_i \in \mathcal{L}$. If $e_{\mathcal{L}} = 2$, then there exists $g \in G$ such that

$$gL_{2i} = \mathfrak{p}^i \oplus \mathfrak{p}^i$$
$$gL_{2i+1} = \mathfrak{p}^i \oplus \mathfrak{p}^{i+1}$$

for all $L_i \in \mathcal{L}$.

Proof. Observe $L_{e_{\mathcal{L}}} = \mathfrak{p}L_0$ by definition. Thus the quotient $L_0/L_{e_{\mathcal{L}}}$ is a 2-dimensional $\mathfrak{k}$-vector space. Then the quotients $L_i/L_{e_{\mathcal{L}}}$ for $0 \leq i \leq e_{\mathcal{L}}$ form a flag of subspaces for the vectorspace $L_0/L_{e_{\mathcal{L}}}$. Therefore by dimensionality $1 \leq e_{\mathcal{L}} \leq 2$.

The $\mathcal{O}$-lattice L_0 is the $\mathcal{O}$-span of an F-basis for V and so there surely exists $g \in G$ such that $gL_0 = \mathcal{O} \oplus \mathcal{O}$. It follows then that $gL_{e_{\mathcal{L}}i} = \mathfrak{p}^i \oplus \mathfrak{p}^i$ for all $i \in \mathbb{Z}$. Thus we are done for $e_{\mathcal{L}} = 1$ and for the first claim when $e_{\mathcal{L}} = 2$. For the second claim when $\rceil_L = 2$, observe that then

$$\mathcal{O} \oplus \mathcal{O} \supseteq gL_1 \supseteq \mathfrak{p} \oplus \mathfrak{p}.$$

The space $U = gL_1/(\mathfrak{p} \oplus \mathfrak{p})$ is a 1-dimensional $\mathfrak{k}$-vector space, so there exists $h \in \mathrm{GL}_2(\mathfrak{k})$ such that $hU = \mathfrak{k} \oplus 0$. Choose a lift $\tilde{h}$ of h in $\mathrm{GL}_2(\mathcal{O})$. Then $\tilde{h}gL_1 = \mathcal{O} \oplus \mathfrak{p}$. The result now follows. □

We associate a ring to a lattice chain $\mathcal{L}$ in V. Define

$$\mathfrak{A}_{\mathcal{L}} = \{x \in A \mid xL_i \subseteq L_i \text{ for all } i \in \mathbb{Z}\} = \bigcap_{i \in \mathbb{Z}} \mathrm{End}_{\mathcal{O}}(L_i).$$

Note that each $L_i \in \mathcal{L}$ is a module over $\mathfrak{A}_{\mathcal{L}}$. We may put the proposition above in terms of the ring $\mathfrak{A}_{\mathcal{L}}$, and see that for any lattice chain $\mathcal{L}$ in V, there exists $g \in G$ such that

$$g\mathfrak{A}_{\mathcal{L}}g^{-1} = \begin{cases} \mathfrak{M} = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{smallmatrix}\right) & \text{if } e_{\mathcal{L}} = 1, \\ \mathfrak{J} = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathfrak{p} \end{smallmatrix}\right) & \text{if } e_{\mathcal{L}} = 2. \end{cases} \tag{3.1}$$

Thus we essentially always work with either $\mathfrak{M}$ or $\mathfrak{J}$ moving foward. If $K = \mathrm{GL}_2(\mathcal{O})$, we note that $\mathfrak{M} = \mathrm{Lie}(K)$ and $\mathfrak{J} = \mathrm{Lie}(I)$, where I is the standard Iwahori as usual.

We define an $\mathfrak{A}_{\mathcal{L}}$-lattice in V to be an $\mathcal{O}$-lattice which is also a module over $\mathfrak{A}_{\mathcal{L}}$.

Proposition 3.4.3. *Let $\mathcal{L}$ be a lattice chain in V and L an $\mathfrak{A}_{\mathcal{L}}$-lattice in V. Then $L \in \mathcal{L}$.*

Proof. See [BH06, §12.2]. □

Thus $\mathcal{L}$ is the set of all $\mathfrak{A}_{\mathcal{L}}$-lattices in V and so $\mathcal{L}$ is recoverable from $\mathfrak{A}_{\mathcal{L}}$. The structure of $\mathfrak{A}_{\mathcal{L}}$ means it is more fruitful to work with $\mathfrak{A}_{\mathcal{L}}$.

Definition 3.4.4. A *chain order* in A is a ring of the form $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$ for some lattice chain $\mathcal{L}$ in V. We use the notation $e_{\mathfrak{A}} = e_{\mathfrak{L}}$.

Definition 3.4.5. A *parahoric subgroup* $P \subseteq G$ is a subgroup which stabilizes a lattice chain $\mathcal{L}$. In other words,

$$P = \{g \in G \mid gL_i = L_i \text{ for all } L_i \in \mathcal{L}\}.$$

Thus the parahoric subgroups of G arise as the invertible elements of the chain orders in A. By (3.1), we see all parahoric subgroups are G-conjugate to either $K = \mathrm{GL}_2(\mathcal{O})$ or the standard Iwahori I. As we primarily work on the level of chain orders in this section, we will often put $\mathfrak{A}^\times$ for the (parahoric) group of invertible elements in a chain order $\mathfrak{A}$.

We define filtrations of $\mathfrak{A}$ and $\mathfrak{A}^\times$, respectively. Begin with $\mathfrak{P} = \operatorname{rad}\mathfrak{A}$, the Jacobson radical. The key observation is that there is a sort of "prime element" of $\mathfrak{A}$, which puts $\mathfrak{A}$ and $\mathfrak{P}$ in a similar relationship as $\mathcal{O}$ is with its maximal ideal $\mathfrak{p}$.

Proposition 3.4.6. *Let $\mathfrak{A}$ be a chain order in A with Jacobson radical $\mathfrak{P}$ and let $e = e_{\mathfrak{A}}$. Then, $\mathfrak{P}^e = \mathfrak{p}\mathfrak{A}$ and there exists $\Pi \in G$ such that*

$$\mathfrak{P} = \Pi\mathfrak{A} = \mathfrak{A}\Pi.$$

Proof. By (3.1) it suffices to check the proposition holds for $\mathfrak{A} = \mathfrak{M}$ and $\mathfrak{A} = \mathfrak{I}$. Let ϖ be a uniformizer of F. Then we have

$$\operatorname{rad}\mathfrak{M} = \varpi\mathfrak{M}$$

and

$$\operatorname{rad}\mathfrak{I} = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \mathfrak{I}.$$

□

We call Π a *prime element* of the chain order $\mathfrak{A}$. One verifies readily that if $\mathfrak{P} = \operatorname{rad}\mathfrak{A}$,

$$\mathfrak{P}^n = \Pi^n\mathfrak{A},$$

for $n \geq 0$. We use this to extend the definition of $\mathfrak{P}^n$ for $n < 0$. If $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ is the lattice chain associated to $\mathfrak{A}$, then $\Pi L_i = L_{i+1}$ and we have

$$\mathfrak{P}^n = \{x \in A \mid xL_i \subseteq L_{i+n}\}.$$

Each $\mathfrak{P}^n$ is a subring of $\mathfrak{A}$ and provides a filtration. We define the filtration on $\mathfrak{A}^\times$ in terms of these $\mathfrak{P}^n$. Set

$$U_{\mathfrak{A}} = U_{\mathfrak{A}}^0 = \mathfrak{A}^\times.$$

Then, for each $n \geq 1$, define

$$U_{\mathfrak{A}}^n = 1 + \mathfrak{P}^n.$$

Each $U_{\mathfrak{A}}^n$ is a normal subgroup of $U_{\mathfrak{A}}$ and a compact open subgroup of G. We also define

$$\mathcal{K}_{\mathfrak{A}} = \{g \in G \mid g\mathfrak{A}g^{-1} = \mathfrak{A}\}.$$

The subgroup $\mathcal{K}_{\mathfrak{A}}$ is the G-normalizer of $U_{\mathfrak{A}}^m$ for all $m \geq 0$. Furthermore it is open in G and contains and is compact modulo the center Z of G.

3.4.2 The Chain Order of a Quadratic Extension

Suppose that $E \subseteq A$ is an F-subalgebra such that E/F is a quadratic field extension. Then there is a unique chain order associated to the extension satsifying certain desiderata.

Proposition 3.4.7. *Let E be as above.*

(i) The set of $\mathcal{O}_E$-lattices in $V = F \oplus F$ is a lattice chain $\mathcal{L}$ with $e_{\mathcal{L}} = e(E/F)$ (the ramification index of E). Furthermore, $\mathcal{L}$ is the unique lattice chain in V stable under $E^\times$-translation.

(ii) The chain order $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$ is the unique chain order in A such that $E^\times \subseteq \mathcal{K}_{\mathfrak{A}}$.

(iii) If $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$, then for $x \in E^\times$, $x\mathfrak{A} = \mathfrak{P}^{v_E(x)}$ and $K_{\mathfrak{A}} = E^\times U_{\mathfrak{A}}$.

Proof. See [BH06, §12.4] □

3.4.3 Chain Orders and Representations

Let ψ be a nontrivial character of F and denote by $\widehat{A}$ the group of characters of A. We define $\psi_A \in \widehat{A}$ by precomposing ψ with the trace map $\mathrm{tr}_A \colon A \to F$. For $a \in A$, we get a character $a\psi_A$ by defining

$$a\psi_A(x) = \psi_A(ax).$$

The map $A \to \widehat{A}$ given by $a \mapsto a\psi_A$ is an isomorphism of groups. We henceforth fix ψ a character of F of level 1 (recall the level of an additive character ψ of F is the least integer n such that ψ is trivial on $\mathfrak{p}^n$). When $2m \geq n > m \geq 1$, then the map $x \mapsto 1 + x$ is an isomorphism

$$\mathfrak{P}^m/\mathfrak{P}^n \to U_{\mathfrak{A}}^m/U_{\mathfrak{A}}^n.$$

With this we immediately have:

Proposition 3.4.8. *Let $\mathfrak{A}$ be a chain order in A with radical $\mathfrak{P}$. Let m, n be integers such that $2m + 1 \geq n > m \geq 0$ and denote by $(U_{\mathfrak{A}}^{m+1}/U_{\mathfrak{A}}^{n+1})^\wedge$ the group of characters of $U_{\mathfrak{A}}^{m+1}/U_{\mathfrak{A}}^{n+1}$. Then*

$$\mathfrak{P}^{-n}/\mathfrak{P}^{-m} \to (U_{\mathfrak{A}}^{m+1}/U_{\mathfrak{A}}^{n+1})^\wedge,$$

given by $a + \mathfrak{P}^{-m} \mapsto \psi_{A,a}|_{U_{\mathfrak{A}}^{m+1}}$ is an isomorphism, where $\psi_{A,a}(x) = \psi_A(a(x-1))$.

Moving forward, we write just ψ_a for $\psi_{A,a}$. From the proposition, we see that we are able to associate characters of the filtration subgroups of $U_{\mathfrak{A}}$ (which are compact open subgroups of G) to cosets of the filtrations of $\mathfrak{A}$. The properties of such characters are thus reflected in properties of the cosets $a + \mathfrak{P}^{-m}$, and hence we can work at the level of matrix computations. In particular, if we take $m = n - 1$, then associated to a coset $a + \mathfrak{P}^{1-n}$ we get a character of $U_{\mathfrak{A}}^n$ which is trivial on $U_{\mathfrak{A}}^{n+1}$. This type of character was key to our first construction of supercuspidal representations, where we had $\mathfrak{A} = \mathfrak{M}$ and $n = 1$.

We now let (π, V) be an irreducible smooth representation of G. Denote by $\mathcal{S}(\pi)$ the set of $(\mathfrak{A}, n)$ where $n \geq 0$ is an integer and $\mathfrak{A}$ is a chain order such that π contains the trivial character of the compact open subgroup $U_{\mathfrak{A}}^{n+1}$.

Definition 3.4.9. The *normalized level* of π, denoted by $\ell(\pi)$, is defined as

$$\ell(\pi) = \min\{n/e_{\mathfrak{A}} \mid (\mathfrak{A}, n) \in \mathcal{S}(\pi)\}.$$

The normalized level is recording in some sense how “deep” we need to look in the parahoric filtrations in G for a trivial character to appear in the representation π. Since $U^1_{\mathfrak{M}} \subseteq U^1_{\mathfrak{J}}$, we have that $\ell(\pi) = 0$ if and only if π contains the trivial character of $U^1_{\mathfrak{M}}$. Observe that this is precisely the situation of theorem 3.3.1. That case gives an insight towards how the appearance of a such trivial character is revealing of the nature of π. We now concern ourselves with those π with $\ell(\pi) > 0$.

3.4.4 Strata

Our primary tool for the analysis of a representation π with $\ell(\pi) > 0$ is *strata*. As before, let $\psi \in \widehat{F}$ be a fixed character of level one.

Definition 3.4.10. A *stratum* in A is a triple $(\mathfrak{A}, n, \alpha)$, where $\mathfrak{A}$ is a chain order in A, $n \in \mathbb{Z}$, and $\alpha \in \mathfrak{P}^{-n}$, where $\mathfrak{P} = \operatorname{rad}\mathfrak{A}$. Two strata $(\mathfrak{A}, n, \alpha_1)$ and $(\mathfrak{A}, n, \alpha_2)$ are *equivalent* if $\alpha_1 \equiv \alpha_2 \pmod{\mathfrak{P}^{1-n}}$.

If $n \geq 1$, then we can associate to a stratum $(\mathfrak{A}, n, \alpha)$ the character ψ_α of $U^n_{\mathfrak{A}}$, trivial on $U^{n+1}_{\mathfrak{A}}$. The definition of equivalence of strata means that this character depends only on the equivalence class of the stratum. We extend the notion of intertwining to strata:

Proposition 3.4.11. *Let $(\mathfrak{A}_1, n_1, \alpha_1)$ and $(\mathfrak{A}_2, n_2, \alpha_2)$ be two strata in A with $n_1, n_2 \geq 1$. Put $\mathfrak{P}_1, \mathfrak{P}_2$ for the respective radicals of the chain orders in each strata and let $g \in G$. Then the following are equivalent.*

(i) g intertwines the character ψ_{α_1} with the character ψ_{α_2}.

(ii) The intersection

$$g^{-1}(\alpha_1 + \mathfrak{P}_1^{1-n_1})g \cap (\alpha_2 + \mathfrak{P}_2^{1-n_2})$$

is nonempty.

Proof. See [BH06, §12.7] □

An element $g \in G$ which satisfies the equivalent conditions of proposition 3.4.11 is said to *intertwine* $(\mathfrak{A}_1, n_1, \alpha_1)$ *with* $(\mathfrak{A}_2, n_2, \alpha_2)$.

3.5 Fundamental Strata

We restrict our focus to a class of strata called *fundamental strata*.

Definition 3.5.1. Let $\mathfrak{A}$ be a chain order in A with radical $\mathfrak{P}$. A stratum $(\mathfrak{A}, n, \alpha)$ is a *fundamental stratum* in A if the coset $\alpha + \mathfrak{P}^{1-n}$ contains no nilpotents.

We remark that this depends only on the equivalence class of the stratum. One shows (cf. [BH06, §12.8]) that any non-fundamental stratum is G-conjugate to one of the following:

$$(\mathfrak{M}, n, \varpi^{-n}\alpha),$$
$$(\mathfrak{J}, 2n-1, \varpi^{-n}\alpha),$$

where $n \in \mathbb{Z}$ and $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

We say that an irreducible representation (π, V) of G *contains the stratum* $(\mathfrak{A}, n, \alpha)$ if the character ψ_α associated to $(\mathfrak{A}, n, \alpha)$ of $U^n_{\mathfrak{A}}$ occurs in π. Recall that the normalized level

$\ell(\pi)$ of π tells us how deep to look in the filtrations of parahorics to find a trivial character occuring in π. If a stratum $(\mathfrak{A}, n, \alpha)$ is contained in π, then since ψ_α is trivial on $U_{\mathfrak{A}}^{n+1}$, surely $\ell(\pi) \leq n/e_{\mathfrak{A}}$. Fundamental strata are particularly important when considering those strata appearing in representations of G. In [Bus87, §3, Theorem 2], it is shown that *every* irreducible smooth representation (π, V) of G with $\ell(\pi) > 0$ contains a fundamental stratum[1]. Moreover, we have the following:

Proposition 3.5.2. *Let (π, V) be an irreducible representation of G with $\ell(\pi) > 0$, so that π contains a fundamental stratum $(\mathfrak{A}, n, \alpha)$. Then $\ell(\pi) = n/e_{\mathfrak{A}}$.*

Proof. See [BH06, §12.9] □

From this there are immediately some things to be said. Firstly, observe that if π is an irreducible smooth representation of G with $0 < \ell(\pi) = n/2 \notin \mathbb{Z}$, then π contains a fundamental stratum (conjugate to one) of the form $(\mathfrak{I}, n, \alpha)$. This follows immediately from [Bus87, §3, Theorem 2] and the proposition above. Similarly, after observing

$$U_{\mathfrak{M}}^{n+1} \subseteq U_{\mathfrak{I}}^{2n+1} \subseteq U_{\mathfrak{I}}^{2n} \subseteq U_{\mathfrak{M}}^{n},$$

we see that if $0 < \ell(\pi) = n \in \mathbb{Z}$, then π contains a fundamental stratum (conjugate to one) of the form $(\mathfrak{M}, n, \alpha)$. Thus we do not need to consider fundamental strata of the form $(\mathfrak{I}, 2n, \alpha)$. In other words,

Proposition 3.5.3. *Let π be an irreducible smooth representation of G such that $\ell(\pi) > 0$. Then π contains a fundamental stratum $(\mathfrak{A}, n, \alpha)$ such that* $\gcd(n, e_{\mathfrak{A}}) = 1$.

3.5.1 Classifying Fundamental Strata

In this section we provide a complete classification of the fundamental strata.

Let $(\mathfrak{A}, n, \alpha)$ be a stratum and suppose that $e_{\mathfrak{A}} = 2$ and n is odd. Then, we say $(\mathfrak{A}, n, \alpha)$ is a *ramified simple stratum.*

There is more granularity when considering $(\mathfrak{A}, n, \alpha)$ with $e_{\mathfrak{A}} = 1$. In this case, we write $\alpha = \varpi^{-n}\alpha_0$ for some $\alpha_0 \in \mathfrak{A}$. We can do this since $\mathfrak{A}$ is G-conjugate to $\mathfrak{M}$. Thus we reduce to this case, whence the radical is $\varpi\mathfrak{M}$ and its powers are given by $\varpi^n\mathfrak{M}$. The element α_0 is some matrix, and we let $f_\alpha(t)$ be its characteristic polynomial. Let $\bar{f}_\alpha(t) \in \mathfrak{k}[t]$ be the reduction of $f_\alpha(t)$ modulo $\mathfrak{p}$, so that $\bar{f}_\alpha(t)$ is the characteristic polynomial of the reduction of α_0 modulo the radical of $\mathfrak{A}$. The polynomial $\bar{f}_\alpha(t)$ depends only on the equivalence class of the stratum. With this, we classify the fundamental stratum based on roots of this polynomial:

Definition 3.5.4. Let $(\mathfrak{A}, n, \alpha)$ be a fundamental stratum such that $e_{\mathfrak{A}} = 1$. If $\bar{f}_\alpha(t)$ is irreducible over $\mathfrak{k}$, then we say the stratum is *unramified simple.* If $\bar{f}_\alpha(t)$ has a repeated root in $\mathfrak{k}^\times$, then we say the stratum is *essentially scalar.* Finally, if $\bar{f}_\alpha(t)$ has distinct roots in $\mathfrak{k}$, then the stratum is *split.*

Remark 6. The case $\bar{f}_\alpha(t) = t^2$ is excluded by the definition of fundamental strata.

Definition 3.5.5. A fundamental stratum is *simple* if it is either ramified simple or unramified simple.

[1]The author in [Bus87] uses a more general definition of fundamental stratum than we use here. Thus the referenced result has no condition on $\ell(\pi)$.

The fundamental strata exhibit quite controlled intertwining properties.

Proposition 3.5.6. *A ramified simple stratum cannot intertwine with any stratum of the form $(\mathfrak{M}, n, \alpha)$. Moreover, if two strata $(\mathfrak{M}, n, \alpha)$ and $(\mathfrak{M}, n, \beta)$ intertwine, then we have $\bar{f}_\alpha(t) = \bar{f}_\beta(t)$.*

Proof. See [BH06, §13.2] □

This proposition allows us to determine in which sorts of representations the different types of fundamental strata may appear.

Theorem 3.5.7. *Suppose π is an irreducible smooth representation of G such that $\ell(\pi) > 0$. Then there is a character χ of $F^\times$ such that $\ell(\chi\pi) < \ell(\pi)$ if and only if π contains an essentially scalar stratum $(\mathfrak{M}, n, \alpha)$.*

Proof. See [BH06, §13.2] □

Combining the previous two results, and recalling proposition 3.2.3, we get:

Corollary 3.5.8. *Let π be an irreducible smooth representation of G with $0 < \ell(\pi) \leq \ell(\chi\pi)$ for all characters χ of $F^\times$. Then exactly one of the following holds:*

(i) π contains a split fundamental stratum.

(ii) π contains a ramified simple stratum.

(iii) π contains an unramified simple stratum.

3.5.2 Fundamental Strata Appearing in the Principal Series

The classification of fundamental strata provides a first step towards a complete classification of the irreducible smooth representations of G. Indeed, we shortly will see that one may distinguish supercuspidal representations from those in the principal series by which kind of fundamental stratum is contained in the representation. In this section we state the results concerning the fundamental strata occuring in the principal series. All proofs may be found in [BH06, §14].

We begin with the split fundamental strata.

Proposition 3.5.9. *Let (π, V) be an irreducible smooth representation of G containing a split fundamental stratum $(\mathfrak{M}, n, \alpha)$. Then, one may choose $\alpha \in T$ and the Jacquet module of π contains the character $\psi_\alpha|_{U^n_{\mathfrak{M}} \cap T}$. In particular, $V_N \neq 0$ and hence π is not supercuspidal.*

If, on the other hand, we begin with a representation in the principal series, we determine the fundamental stratum appearing in the representation easily.

Proposition 3.5.10. *Suppose $\chi = \chi_1 \otimes \chi_2$ is a character of T inflated to one of B, and put $(\Sigma, X) = \mathrm{Ind}_B^G \chi$. Let n_1 and n_2 be the levels of χ_1 and χ_2, respectively (recall for a multiplicative character χ, the level is the least integer n such that χ is trivial on U_F^{n+1}). Then:*

(i) If $n = \max(n_1, n_2) > 0$ and $\chi_1\chi_2^{-1}|_{U_F^n} \neq 1$, then Σ contains a fundamental split stratum.

(ii) If $n_1 = n_2 = n \neq 0$ and $\chi_1\chi_2^{-1}|_{U_F^n} = 1$, then Σ contains a fundamental essentially scalar stratum.

(iii) If $n_1 = n_2 = 0$, then Σ contains the trivial character of $U_{\mathfrak{J}}^1$.

Let π be an irreducible smooth representation containing the trivial character of $U_{\mathfrak{J}}^1$ and with $\ell(\pi) > 0$. Thus we are in case (iii) of the above proposition. Observe that case (ii) of theorem 3.3.1 also puts us in this situation. We show such a representation is necessarily not supercuspidal.

Proposition 3.5.11. *Let (π, V) be an irreducible smooth representation of G containing a character φ of $I = U_{\mathfrak{J}}$ which is trivial on $U_{\mathfrak{J}}^1$. Then the projection $V \to V_N$ is injective on the isotypic subspace V^φ. In particular, π is not supercuspidal.*

The main consequence of these results is the following:

Theorem 3.5.12 (Exhaustion Theorem)**.** *Let (π, V) be an irreducible smooth representation of G such that $\ell(\pi) \leq \ell(\chi\pi)$ for all characters χ of $F^\times$. Then the following are equivalent:*

(i) π is supercuspidal.

(ii) Either

(a) $\ell(\pi) = 0$ and π contains a representation of $U_{\mathfrak{M}}$ inflated from an irreducible cuspidal representation of $\mathrm{GL}_2(\mathfrak{k})$, or

(b) $\ell(\pi) > 0$ and π contains a fundamental simple stratum.

Proof. If $\ell(\pi) = 0$, then proposition 3.5.11 and theorem 3.3.1 give the result. Suppose $\ell(\pi) > 0$. If $\ell(\pi)$ does not contain a simple stratum then by hypothesis it must contain a split fundamental stratum. This implies that π is not supercuspidal, thus (i) implies (ii).

Now suppose that π is not supercuspidal. Thus we may identify π with a G-subspace of $\Sigma = \mathrm{Ind}_B^G \chi$ where $\chi = \chi_1 \otimes \chi_2$ is a character of T. Assume first that Σ is irreducible and so $\pi = \Sigma$. If χ_1 or χ_2 has level at least 1, then Σ contains either a fundamental split or essentially scalar stratum. The hypothesis on $\ell(\pi)$ excludes the latter. Thus π contains a fundamental split stratum, meaning π cannot contain a simple stratum.

If both χ_1 and χ_2 have level 0, then $\pi = \Sigma$ contains the trivial character of $U_{\mathfrak{J}}^1$, which means $\ell(\pi) = 0$, which has already been dealt with. Thus it only remains to check when Σ is reducible.

If Σ is reducible, then π is either of the form $\varphi \circ \det$ or $\varphi \cdot \mathrm{St}_G$ for some character φ of $F^\times$. In either case, normalized level $\ell(\pi)$ is equal to the level of φ. The minimality hypothesis on $\ell(\pi)$ then requires that the level of φ is 0. The result then follows. □

3.5.3 Simple Strata

The Exhaustion Theorem (3.5.12) shows that for understanding a supercuspidal representation π (when $\ell(\pi)$ satisfies a minimality condition) it will be helpful to understand simple strata. In this section we collect some results to see how simple strata arise naturally from certain quadratic extensions of F.

Definition 3.5.13. Let $\alpha \in G \setminus Z$. We say α is *minimal over F* if the subalgebra $E = F[\alpha] \subseteq A$ is a field and, if $n = -v_E(\alpha)$, one of the following holds:

(i) The extension E/F is totally ramified and n is odd, or

(ii) The extension E/F is unramified, and the coset $\varpi^n\alpha + \mathfrak{p}_E$ generates the field extension $\mathfrak{k}_E/\mathfrak{k}_F$.

Since $\alpha \in G \setminus Z$ we have that E/F is a quadratic field extension. Furthermore, $f(E/F)v_E(\alpha) = v_F(\det\alpha)$, where $f(E/F)$ is the residue degree of E/F. When an extension E/F is generated by a minimal element, then it is easy to describe the ring of integers in E.

Lemma 3.5.14. *Let $\alpha \in G$ be minimal over F and let $E = F[\alpha]$, $n = -v_E(\alpha)$. Define*

$$\alpha_0 = \begin{cases} \varpi^{(n+1)/2}\alpha & \text{if } E/F \text{ is ramified,} \\ \varpi^n\alpha & \text{if } E/F \text{ is unramified.} \end{cases}$$

Then $\mathcal{O}_E = \mathcal{O}_F[\alpha_0]$.

Simple strata give rise to minimal elements.

Proposition 3.5.15. *Suppose that $(\mathfrak{A}, n, \alpha)$ is a simple stratum in A. Then the following hold:*

(i) The element α is minimal over F.

(ii) $F[\alpha]^\times \subseteq \mathcal{K}_\mathfrak{A}$.

(iii) $e(F[\alpha]/F) = e_\mathfrak{A}$.

(iv) every $\alpha' \in \alpha + \mathfrak{P}^{1-n}$ is minimal over F, where $\mathfrak{P} = \mathrm{rad}\,\mathfrak{A}$.

A result in the converse direction also holds.

Proposition 3.5.16. *Let $\alpha \in G$ be minimal over F. Then there exists a unique chain order $\mathfrak{A}$ in A such that $\alpha \in \mathcal{K}_\mathfrak{A}$. Moreover, $F[\alpha]^\times \subseteq \mathcal{K}_\mathfrak{A}$. If we put $n = -v_{F[\alpha]}(\alpha)$, then the triple $(\mathfrak{A}, n, \alpha)$ is a simple stratum.*

Both propositions are proved in [BH06, §13.4-13.5].

3.6 The Classification of Supercuspidal Representations

With the results of the previous section, we are equipped to proceed in giving a complete classification of the supercuspidal representations of $G = \mathrm{GL}_2(F)$. We fix a character ψ of F of level one.

Let $(\mathfrak{A}, n, \alpha)$ be a simple stratum, so that $E = F[\alpha]$ is a degree 2 extension of F with $E^\times \subseteq \mathcal{K}_\mathfrak{A}$. From the element α, using proposition 3.4.8 with $m = [n/2]$ (here the brackets $[\cdot]$ denote the integer part) we get the character ψ_α of $U_\mathfrak{A}^{[n/2]+1}$ trivial on $U_\mathfrak{A}^{n+1}$. We collect two key results, whose proofs may be found in [BH06, §16].

Theorem 3.6.1 (Intertwining Theorem)**.** *Let $(\mathfrak{A}, n, \alpha)$, $n \geq 1$, be a simple stratum and let $E = F[\alpha]$. Let $g \in G$. The following are equivalent:*

(i) The element g intertwines the character ψ_α of $U_\mathfrak{A}^{[n/2]+1}$.

(ii) The element g normalizes the character ψ_α of $U_{\mathfrak{A}}^{[n/2]+1}$.

(iii) $g \in E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$.

Theorem 3.6.2 (Conjugacy Theorem)**.** *Let $(\mathfrak{A}_1, n, \alpha_1)$ and $(\mathfrak{A}_2, n, \alpha_2)$ be two simple strata, with $n \geq 1$. The characters ψ_{α_1} and ψ_{α_2} of $U_{\mathfrak{A}}^{[n/2]+1}$ intertwine in G if and only if they are conjugate by an element of $U_{\mathfrak{A}}$.*

The group appearing in (iii) of the Intertwining Theorem 3.6.1 plays a critical role. So critical that we give it its own letter: Let $J_\alpha = E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$. The group J_α contained in $\mathcal{K}_{\mathfrak{A}}$ and is open in G. It contains and is compact modulo the center Z of G. We immediately can see its importance:

Theorem 3.6.3. *Keeping with the notation we have used thus far, let Λ be an irreducible representation of J_α which contains the character ψ_α of $U_{\mathfrak{A}}^{[n/2]+1}$. Then the restriction $\Lambda|_{U_{\mathfrak{A}}^{[n/2]+1}}$ is a multiple of ψ_α. Moreover, the representation*

$$\pi_\Lambda = \text{c-Ind}_{J_\alpha}^G \Lambda$$

is irreducible and supercuspidal.

Proof. If we restrict Λ to $U_{\mathfrak{A}}^{[n/2]+1}$ then we get a direct sum of irreducible representations, all of which are J_α-conjugate. One of these is ψ_A, which is normalized (cf. Conjugacy Theorem and Intertwining Theorem) by J_α, implying $\Lambda|_{U_{\mathfrak{A}}^{[n/2]+1}}$ is a multiple of ψ_A. Then if g intertwines Λ it must also intertwine ψ_α, whence $g \in J_\alpha$ by the Intertwining Theorem. The result then follows from theorem 3.2.4. □

We introduce notation to record representations that occur in this way. For a simple stratum $(\mathfrak{A}, n, \alpha)$ with $n \geq 1$, we let $C(\psi_\alpha, \mathfrak{A})$ denote the set of equivalence classes of irreducible representations Λ of J_α such that $\Lambda|_{U_{\mathfrak{A}}^{[n/2]+1}}$ is a multiple of ψ_α. The sets $C(\psi_\alpha, \mathfrak{A})$ for various simple strata in A are essentially unique up to G-conjugacy when the representations therewithin induce equivalent representations of G. More precisely (cf. [BH06, §15.4]):

Theorem 3.6.4. *Let $(\mathfrak{A}_1, n_1, \alpha_1)$ and $(\mathfrak{A}_2, n_2, \alpha_2)$ be simple strata in A with $n_1, n_2 \geq 1$. Let $\Lambda_1 \in C(\psi_{\alpha_1}, \mathfrak{A}_1)$ and $\Lambda_2 \in C(\psi_{\alpha_2}, \mathfrak{A}_2)$ and suppose that $\pi_{\Lambda_1} \cong \pi_{\Lambda_2}$ (notation of theorem 3.6.3). Then, $n_1 = n_2$ and there exists $g \in G$ such that $\mathfrak{A}_2 = g^{-1}\mathfrak{A}_1 g$, $J_{\alpha_2} = g^{-1}J_{\alpha_1}g$, and $\Lambda_2 = \Lambda_1^g$. When $\mathfrak{A}_1 = \mathfrak{A}_2$, then g may be chosen in $U_{\mathfrak{A}_1}$.*

3.6.1 Cuspidal Types

Definition 3.6.5. A *cuspidal type in G* is a triple $(\mathfrak{A}, J, \Lambda)$ where $\mathfrak{A}$ is a chain order in A, $J \subseteq \mathcal{K}_{\mathfrak{A}}$ is a subgroup, and Λ is an irreducible smooth representation of J, occuring in one of the following ways:

(i) We have $\mathfrak{A} \cong \mathfrak{M}$, $J = ZU_{\mathfrak{A}}$, and $\Lambda|_{U_{\mathfrak{A}}}$ is the inflation of an irreducible cuspidal representation of $U_{\mathfrak{A}}/U_{\mathfrak{A}}^1 \cong \mathrm{GL}_2(\mathfrak{k})$.

(ii) There is a simple stratum $(\mathfrak{A}, n, \alpha)$, with $n \geq 1$, such that $J = J_\alpha$ and $\Lambda \in C(\psi_\alpha, \mathfrak{A})$.

(iii) There is a triple $(\mathfrak{A}, J, \Lambda_0)$ satisfying (i) or (ii) and a character χ of $F^\times$ such that $\Lambda \cong \Lambda_0 \otimes \chi \circ \det$.

Observe that by theorems 3.3.1 and 3.6.3, for any cupsidal type $(\mathfrak{A}, J, \Lambda)$ we have that $\text{c-Ind}_J^G \Lambda$ is an irreducible supercuspidal representation of G. The main result of this section allows us to go the other way:

Theorem 3.6.6 (Induction Theorem). *Let π be an irreducible supercuspidal representation of G. There exists a cuspidal type $(\mathfrak{A}, J, \Lambda)$ in G such that $\pi \cong \text{c-Ind}_J^G \Lambda$. The cuspidal type $(\mathfrak{A}, J, \Lambda)$ is determined by π uniquely up to G-conjugacy.*

Proof. To begin, observe in case (iii) of definition 3.6.5, we have

$$\text{c-Ind}_J^G(\Lambda_0 \otimes \chi \circ \det) = \chi \cdot \text{c-Ind}_J^G(\Lambda_0),$$

so it suffices to show the theorem for cases (i) and (ii) of definition 3.6.5. If $\ell(\pi) = 0$, then the theorem follows directly from theorem 3.5.12. Thus we assume $\ell(\pi) > 0$.

Here, theorem 3.5.12 gives that there exists a simple stratum $(\mathfrak{A}, n, \alpha)$, with $n \geq 1$, such that π contains the character ψ_α of $U_{\mathfrak{A}}^n$. It follows that π contains a character ξ of $U_{\mathfrak{A}}^{[n/2]+1}$ such that $\xi|_{U_{\mathfrak{A}}^n} = \psi_\alpha$. This character ξ is of the form ψ_β for some $\beta \cong \alpha \pmod{\mathfrak{P}^{1-n}}$, where as usual $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$. The results of section 3.5.3 give that then $(\mathfrak{A}, n, \beta)$ is also a simple stratum. The representation π then contains a representation Λ of J_β such that $\Lambda|_{U_{\mathfrak{A}}^{[n/2]+1}}$ contains ψ_β, whence $\Lambda|_{U_{\mathfrak{A}}^{[n/2]+1}}$ is a multiple of ψ_β. Therefore the triple $(\mathfrak{A}, J_\beta, \Lambda)$ is a cuspidal type in G. From Frobenius reciprocity we have

$$0 \neq \operatorname{Hom}_{J_\beta}(\Lambda, \operatorname{Res}_{J_\beta}^G \pi) \cong \operatorname{Hom}_G(\text{c-Ind}_{J_\beta}^G \Lambda, \pi).$$

As $\text{c-Ind}_{J_\beta}^G \Lambda$ is irreducible, we conclude $\text{c-Ind}_{J_\beta}^G \Lambda \cong \pi$. The unicity claim follows from theorem 3.6.4. □

Corollary 3.6.7. *The map*

$$(\mathfrak{A}, J, \Lambda) \mapsto \text{c-Ind}_J^G \Lambda = \pi_\Lambda$$

induces a bijection from the G-conjugacy classes of cuspidal types in G to the set of equivalence classes of irreducible supercuspidal representations of G.

We have arrived at a complete classification of the supercuspidal representations of G in terms of cuspidal types. The issue of supercuspidal representations, then, is reduced to the study of cuspidal types. In definition 3.6.5, case (i) is completely understood. Thus we focus on case (ii).

In this setting, to understand a cuspidal type $(\mathfrak{A}, J, \Lambda)$ we must understand the representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ of $J = J_\alpha = E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$, where $(\mathfrak{A}, n, \alpha)$ is a simple stratum with $n \geq 1$ and $E = F[\alpha]$.

The following subgroups will be necessary: First we define

$$H_\alpha^1 = U_E^1 U_{\mathfrak{A}}^{[n/2]+1}.$$

We also have

$$J_\alpha^1 = J_\alpha \cap U_{\mathfrak{A}}^1 = U_E^1 U_{\mathfrak{A}}^{[(n+1)/2]}.$$

Remark 7. The groups J_α^1 and H_α^1 are the same if and only if n is odd.

First we deal with when n is odd.

Proposition 3.6.8. *Suppose that n is odd. Then every $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ has dimension 1 and two characters $\Lambda_1, \Lambda_2 \in C(\psi_\alpha, \mathfrak{A})$ intertwine in G if and only if $\Lambda_1 = \Lambda_2$.*

Proof. We have that $E^\times \cap U_{\mathfrak{A}}^{[n/2]+1} = U_E^{[n/2]+1}$. If φ is a character of $E^\times$ which agrees with ψ_α on $U_E^{[n/2]+1}$, then we may define a character Λ of J_α by $\Lambda(ux) = \psi_\alpha(u)\varphi(x)$, for $x \in E^\times$, $u \in U_{\mathfrak{A}}^{[n/2]+1}$. Every irreducible representation of J_α containing ψ_α occurs in this way, whence the first claim. To show the claim on intertwining, if $g \in G$ intertwines Λ_1 with Λ_2, then it must intertwine ψ_α and hence lies in J_α by theorem 3.6.1. Thus the g fixes each Λ_i and the result follows. □

When n is even, then there is more subtlety in the situation. If n is even then E/F is unramified and we have $\mathfrak{A} \cong \mathfrak{M}$. Similar to the proof above, one may extend the character ψ_α to one of H_α^1. A lemma proven in [BH06, §16.4] ensures that for such a character θ of H_α^1 extending ψ_α, there exists a uniquely determined irreducible representation η_θ of J_α^1 with the properties that $\eta_\theta|_{H_\alpha^1}$ is a multiple of θ and $\dim \eta_\theta = q$. With this, we have the following.

Proposition 3.6.9. *Keeping with the notation used thus far,*

(i) An element $g \in G$ intertwines η_θ if and only if $g \in J_\alpha$.

(ii) The representation η_θ admits extension to an irreducible representation of J_α lying in $C(\psi_\alpha, \mathfrak{A})$.

(iii) Any representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ satisfies $\Lambda|_{J_\alpha^1} \cong \eta_\theta$ for a uniquely determined character θ of H_α^1.

(iv) Two representations $\Lambda_1, \Lambda_2 \in C(\psi_\alpha, \mathfrak{A})$ intertwine in G if and only if they are equivalent.

To end this section, we introduce an alternative way to track the same data.

Definition 3.6.10. A *cuspidal inducing datum in G* is a pair $(\mathfrak{A}, \Xi)$, where $\mathfrak{A}$ is a chain order in A and Ξ is an irreducible smooth representation of $\mathcal{K}_{\mathfrak{A}}$ of the form $\Xi = \text{c-Ind}_J^{\mathcal{K}_{\mathfrak{A}}} \Lambda$ for some cuspidal type $(\mathfrak{A}, J, \Lambda)$.

The transitivity of induction of representations permits us to restate some of the results regarding cuspidal types in terms of cuspidal inducing data:

Proposition 3.6.11. *Let $(\mathfrak{A}, \Xi)$ be a cuspidal inducing datum in G. Then the representation $\pi_\Xi = \text{c-Ind}_{\mathcal{K}_{\mathfrak{A}}}^G \Xi$ is an irreducible supercuspidal representation of G. Moreover, the map $(\mathfrak{A}, \Xi) \mapsto \pi_\Xi$ is a bijection from the set of G-conjugacy classes of cuspidal inducing data to the set of equivalence classes of irreducible supercuspidal representations of G.*

3.7 Tame Parametrization

When constructing the Langlands correspodence, we will handle the supercuspidal representations of $G = \mathrm{GL}_2(F)$ in terms of *admissible pairs*, which are pairs $(E/F, \chi)$ consisting of a tamely ramified quadratic extension E/F and a character χ of $E^\times$ subject to some conditions on factoring through the field norm map $N_{E/F}$. To each admissible pair we are able to attach an irreducible supercuspidal representation. The main result of this section, the Tame Parametrization Theorem, will show that the association of an irreducible supercuspidal representation to an admissible pair is a bijection between the isomorphism classes of admissible pairs and the equivalence classes of irreducible supercuspidal representations of G.

Remark 8. In this thesis we are always assuming the residual characteristic of F is $p \neq 2$. Thus, any quadratic extension of F is tamely ramified.

Definition 3.7.1. Let $(E/F, \chi)$ be a pair consisting of a tamely ramified extension E/F and a character χ of $E^\times$. The pair $(E/F, \chi)$ is an *admissible pair* if

(i) The character χ does not factor through the norm map $N_{E/F} \colon E^\times \to F^\times$.

(ii) If $\chi|_{U_E^1}$ factors through $N_{E/F}$, then E/F is unramified.

If we let n be the level of χ, then the admissible pair $(E/F, \chi)$ is said to be *minimal* if $\chi|_{U_E^n}$ does not factor through $N_{E/F}$. Two admissible pairs $(E/F, \chi)$ and $(E'/F, \chi')$ are *F-isomorphic* (or just *isomorphic*) if there exists an F-isomorphism $j \colon E \to E'$ such that $\chi = \chi' \circ j$.

Remark 9. If $(E/F, \chi)$ and $(E/F, \chi')$ are two admissible pairs with the same tamely ramified extension E/F in each pair, then the condition of being F-isomorphic amounts to $\chi' = \chi \circ \sigma$ for some $\sigma \in \mathrm{Gal}(E/F)$.

Remark 10. For a character φ of $F^\times$, let $\varphi_E = \varphi \circ N_{E/F}$. If $(E/F, \chi)$ is an admissible pair, then so too is $(E/F, \chi \otimes \varphi_E)$. With this, any admissible pair $(E/F, \chi)$ is F-isomorphic to an admissible pair of the form $(E/F, \chi' \otimes \varphi_E)$ where $(E/F, \chi')$ is a minimal admissible pair.

3.7.1 Attaching Representations to Admissible Pairs

Let $(E/F, \chi)$ be an admissible pair with χ being of level zero. This means that χ is trivial on U_E^1, which, from the definition of admissible pairs, requires that the extension E/F is unramified. We will begin by associating an irreducible supercuspidal G-representation to such an admissible pair.

Lemma 3.7.2. *Suppose E/F is an unramified quadratic extension and χ is a character of $E^\times$ of level zero. Let σ be the nontrivial element of* $\mathrm{Gal}(E/F)$*. Then the following are equivalent.*

(i) The pair $(E/F, \chi)$ is admissible.

(ii) $\chi \neq \chi \circ \sigma$.

(iii) $\chi|_{U_E} \neq (\chi \circ \sigma)|_{U_E}$.

Proof. The equivalence of (ii) and (iii) is immediate upon the observation that, as E/F is unramified, $E^\times = F^\times U_E$. The kernel of $N_{E/F}$ is $\{\sigma(x)/x \mid x \in E^\times\}$. Therefore χ factors through $N_{E/F}$ if and only if $\chi = \chi \circ \sigma$. Since E/F is unramified by assumption, the factoring condition is all that is needed to check if the pair $(E/F, \chi)$ is admissible, and so (i) and (ii) are equivalent. ☐

To proceed, choose an F-embedding of E in $A = \mathrm{Mat}_2(F)$. Recall (proposition 3.4.7) that there is a unique chain order $\mathfrak{A}$ with $E^\times \subseteq \mathcal{K}_\mathfrak{A}$. Conjugating if necessary, we may take $\mathfrak{A} = \mathfrak{M}$. Then, we get an embedding of $\mathcal{O}_E$ in $\mathfrak{M}$, which, in turn, gives a $\mathfrak{k}_F$-embedding of $\mathfrak{k}_E$ in $\mathrm{Mat}_2(\mathfrak{k}_F)$. The supercuspidal representation attached to $(E/F, \chi)$ is now constructed as follows:

The restriction $\chi|_{U_E}$ is the inflation of a regular (cf. (iii) in the lemma and §1.3.3) character $\tilde{\chi}$ of $\mathfrak{k}_E^\times$. The constructions of §1.3.3 then yield from $\tilde{\chi}$ an irreducible cuspidal representation $\tilde{\lambda}$ of $\mathrm{GL}_2(\mathfrak{k}_F)$. We then inflate $\tilde{\lambda}$ to a representation λ of $\mathrm{GL}_2(\mathcal{O}_F) = U_\mathfrak{M}$, whose restriction to U_F is a multiple of $\chi|_{U_F}$. Thus we may extend λ to a representation Λ of $F^\times U_\mathfrak{M} = \mathcal{K}_\mathfrak{M}$ by demanding $\Lambda|_{F^\times}$ be a multiple of $\chi_{F^\times}$. Then triple $(\mathfrak{M}, \mathcal{K}_\mathfrak{M}, \Lambda)$ is a cuspidal type, and so we get the irreducible supercuspidal representation

$$\pi_\chi = \text{c-Ind}_{\mathcal{K}_\mathfrak{M}}^G \Lambda.$$

It is not too much work to verify the supercuspidal representation π_χ depends only the isomorphism class of the admissible pair $(E/F, \chi)$. The π_χ has normalized level $\ell(\pi_\chi) = 0$.

If $(E/F, \chi)$ is an admissible pair with χ of level 0, then the process $(E/F, \chi) \mapsto \pi_\chi$ is a bijection (cf. [BH06, §19.1]) from the set of isomorphism classes of admissible pairs $(E/F, \chi)$, χ of level 0, which we denote by $\mathbb{P}_2(F)_0$, and the set of irreducible supercuspidal G-representations π with $\ell(\pi) = 0$, denoted $\mathscr{A}_2^0(F)_0$. The bijection satisfies desirable properties, all of which follow readily from the construction of π_χ:

Proposition 3.7.3. *The map $(E/F, \chi) \mapsto \pi_\chi$ is a bijection $\mathbb{P}_2(F)_0 \to \mathscr{A}_2^0(F)_0$. If $(E/F, \chi) \in \mathbb{P}_2(F)_0$, then*

(i) If φ is a character of $F^\times$ of level 0, then $\pi_{\chi\varphi_E} = \varphi\pi_\chi$.

(ii) The central character ω_{π_χ} of π_χ is $\omega_{\pi_\chi} = \chi|_{F^\times}$.

(iii) The pair $(E/F, \check{\chi})$ is admissible and $\check{\pi}_\chi = \pi_{\check{\chi}}$.

Next we show how to attach an irreducible supercuspidal representation to a minimal admissible pair $(E/F, \chi)$ when the level of χ is greater than 0. We detail the construction, but for the proofs some propositions upon which it relies we direct the reader to [BH06, §19.4].

Fix ψ an additive character of F of level 1. Let $(E/F, \chi)$ be a minimal admissible pair with χ of level $n \geq 1$. We use the notation $\psi_E = \psi \circ \mathrm{tr}_{E/F}$, and $\psi_A = \psi \circ \mathrm{tr}_A$. Choose an element $\alpha \in \mathfrak{p}_E^{-n}$ such that $\chi(1 + x) = \psi_E(\alpha x)$ for all $x \in \mathfrak{p}_E^{[n/2]+1}$. The element α is necessarily minimal over F (cf. [BH06, §18.1-18.2]). Choose an F-embedding of E into A and let $\mathfrak{A}$ be the unique chain order in A with $E^\times \subseteq \mathcal{K}_\mathfrak{A}$. Then $e_\mathfrak{A} = e(E/F)$ and $(\mathfrak{A}, n, \alpha)$ is a simple stratum.

Using the notation of the previous sections, associated the stratum $(\mathfrak{A}, n, \alpha)$ we have the subgroups J_α, J_α^1, and H_α^1. We construct a $\Lambda \in C(\psi_\alpha, \mathfrak{A})$. The process will feel familiar to the analysis of cuspidal types in § 3.6.1.

We begin with the easier case, and suppose that $n = 2m + 1$ is odd. Then $J_\alpha = E^\times U_\mathfrak{A}^{m+1}$. We define $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ as the character of J_α given by $\Lambda|_{E^\times} = \chi$ and $\Lambda|_{U_\mathfrak{A}^{m+1}} = \psi_\alpha$. The triple $(\mathfrak{A}, J_\alpha, \Lambda)$ is then a cuspidal type and so

$$\pi_\chi = \text{c-Ind}_{J_\alpha}^G \Lambda$$

is an irreducible supercuspidal representation containing the fundamental stratum $(\mathfrak{A}, n, \alpha)$. Therefore $\ell(\pi_\chi) = n/e(E/F)$ and the central character is $\omega_{\pi_\chi} = \chi|_{F^\times}$.

Now suppose that the level of χ is $n = 2m > 0$. Note that this implies χ is trivial on U_E^1 and thus, by definition, E/F is unramified. We define θ a character of $H_\alpha^1 = U_E^1 U_{\mathfrak{A}}^{m+1}$ by

$$\theta(ux) = \psi_\alpha(u)\chi(x),$$

for $u \in U_{\mathfrak{A}}^{m+1}$ and $x \in U_E^1$. Then, as in § 3.6.1, we put $\eta = \eta_\theta$ for the unique representation of $J_\alpha^1 = U_E^1 U_{\mathfrak{A}}^m$ which contains θ. The key result is the following:

Proposition 3.7.4. *There is a unique irreducible representation Λ of J_α such that*

(i) $\Lambda|_{J_\alpha^1} \cong \eta$.

(ii) $\Lambda|_{F^\times}$ *is a multiple of* $\chi|_{F^\times}$.

(iii) For every $\zeta \in \mu_E \setminus \mu_F$ (μ_F denotes the roots of unity in F) we have $\operatorname{tr}\Lambda(\zeta) = -\chi(\zeta)$.

The representation Λ of the proposition above ends up lying in $C(\psi_\alpha, \mathfrak{A})$. We set

$$\pi_\chi = \text{c-Ind}_{J_\alpha}^G \Lambda,$$

in this case, to yield an irreducible supercuspidal representation of G. Here, we have $\ell(\pi_\chi) = n$ and the central character is once more $\omega_{\pi_\chi} = \chi|_{F^\times}$.

In all constructions above, the π_χ that is produced does not depend on any choices made (that of ψ, of α, and of the embedding $E \to A$). Moreover, if φ is a character of $F^\times$ such that $(E/F, \chi\varphi_E)$ is minimal, then one has (cf. [BH06, §19.5])

$$\pi_{\chi\varphi_E} = \varphi\pi_\chi.$$

If $(E/F, \chi)$ is any admissible pair, then there exists χ' such that $(E/F, \chi')$ is a minimal admissible pair and φ a character of $F^\times$ such that $\chi = \chi'\varphi_E$. We define $\pi_\chi = \varphi\pi_{\chi'}$. This construction is independent of choice of χ' and φ. Then $\ell(\pi_\chi) = n/e(E/F)$ where n is the level of χ, and $\omega_{\pi_\chi} = \chi|_{F^\times}$.

In every case detailed in the paragraphs preceding, the π_χ produced depends only on the isomorphism class of the admissible pair $(E/F, \chi)$. Let $\mathbb{P}_2(F)$ be the set of such isomorphism classes, and $\mathscr{A}_2^0(F)$ the set of equivalence classes of irreducible supercuspidal representations of G. The main result is the following.

Theorem 3.7.5 (Tame Parametrization Theorem)**.** *The map $(E/F, \chi) \mapsto \pi_\chi$ is a bijection*

$$\mathbb{P}_2(F) \to \mathscr{A}_2^0(F).$$

For $(E/F, \chi) \in \mathbb{P}_2(F)$, we have:

(i) If χ has level n, then $\ell(\pi_\chi) = n/e(E/F)$.

(ii) The central character of π_χ is given by $\omega_{\pi_\chi} = \chi|_{F^\times}$.

(iii) The pair $(E/F, \check{\chi})$ is also admissible, and $\pi_{\check{\chi}} = \check{\pi}_\chi$.

(iv) If φ is a character of $F^\times$, then $\pi_{\chi\varphi_E} = \varphi\pi_\chi$.

Proof. Observe that the properties listed in the theorem have been observed already. The full details of the map being a bijection are given in [BH06, §20 - §22]. □

Remark 11. Here is an instance where our assumption on the residual characteristic of F is pivotal. If $\operatorname{char}(\mathfrak{k}_F) = 2$, then the map in the theorem above is *not* a bijection. Instead, it is a bijection onto the set of equivalence classes of *unramified* irreducible supercuspidal representations of G. An irreducible supercuspidal representation π of G is unramified if there exists a nontrivial unramified character φ of $F^\times$ such that $\varphi\pi = \pi$.

Chapter 4

Weil Representations

Having completed our study of the representation theory of $\mathrm{GL}_2(F)$, we shift our attention to the Galois side of the picture. We start with recalling the definition of the *Weil group* of F, $\mathcal{W}_F$, some of its basic properties, and how those properties manifest in its representations and Artin's L-functions attached to such representations.

It turns out that for the local Langlands correspondence to hold, one needs to consider a certain class of Weil representations, called *Weil-Deligne representations*. We will see how these naturally arise when considering ℓ-adic representations of $\mathcal{W}_F$ (representations over $\overline{\mathbb{Q}_\ell}$ for $\ell \neq p$).

4.1 The Weil Group

We first recall the definition of $\mathcal{W}_F$. Fix a separable algebraic closure F^{sep} of F and denote by G_F the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. For each positive integer m, F admits a unique unramified extension $F_m \subseteq F^{\mathrm{sep}}$ of degree m. One has $\mathrm{Gal}(F_m/F) = \mathbb{Z}/m\mathbb{Z}$, which is generated by the element φ_m which acts on the residue field as $\varphi_m \colon x \mapsto x^q$. We put $\Phi_m = \varphi_m^{-1}$. Let F^{nr} the maximal unramified extension of F in F^{sep}. Then,

$$\mathrm{Gal}(F^{\mathrm{nr}}/F) = \varprojlim_{m\geq 1} \mathrm{Gal}(F_m/F) = \varprojlim_{m\geq 1} \mathbb{Z}/m\mathbb{Z} = \widehat{\mathbb{Z}}.$$

Then, there is a unique element $\Phi_F \in \mathrm{Gal}(F^{\mathrm{nr}}/F)$ such that Φ_F acts as Φ_m on F_m for all m. Any element of G_F whose image under the quotient map

$$G_F \to \mathrm{Gal}(F^{\mathrm{nr}}/F)$$

Φ_F is said to be a *(geometric) Frobenius element.* The kernel of the above quotient map is the intertia subgroup $I_F = \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{nr}})$.

Inside $\mathrm{Gal}(F^{\mathrm{nr}}/F)$, the element Φ_F generates an infinite cyclic subgroup $\langle \Phi_F \rangle \cong \mathbb{Z}$. The *Weil group of* F (relative to F^{sep}), $\mathcal{W}_F$, is, as an abstract group, the preimage of $\langle \Phi_F \rangle$ in G_F. However, $\mathcal{W}_F$ is topologized in a way that does *not* agree with the subspace topology it inherits from the profinite topology on G_F. Instead, $\mathcal{W}_F$ is topologized with the unique topology such that $I_F \subseteq \mathcal{W}_F$ is open and the subspace topology on I_F in $\mathcal{W}_F$ agrees with that inhertied by I_F from G_F.

If E/F is a finite extension in F^{sep}, then the natural inclusion $G_E \to G_F$ induces a isomorphism (both topological and algebraic) of $\mathcal{W}_E$ onto an open subgroup of $\mathcal{W}_F$, which is normal if and only if E/F is Galois. We identify this subgroup with $\mathcal{W}_E$ henceforth. For a more robust account of the Weil group, [Tat79] is an excellent soruce.

4.1.1 Representations of $\mathcal{W}_F$

Since G_F is a profinite group, its representations are all semisimple. On the other hand, $\mathcal{W}_F$ is locally profinite, and admits $\mathbb{Z}$ as a quotient. Thus, as $\mathrm{Rep}(\mathbb{Z})$ is not semisimple, there exist indecomposable smooth representations of $\mathcal{W}_F$. In this section, we consider smooth representations of $\mathcal{W}_F$ over $\mathbb{C}$.

In the irreducible case, the representation theory of $\mathcal{W}_F$ is closely tied to that of G_F. Since $\mathcal{W}_F$ embeds in G_F with dense image, the category $\mathrm{Rep}(G_F)$ may be indentified with a subcategory of $\mathrm{Rep}(\mathcal{W}_F)$. A smooth representation of $\mathcal{W}_F$ is said to be of *Galois type* if it lies in $\mathrm{Rep}(G_F)$. More concretely, let $\iota\colon \mathcal{W}_F \to G_F$ be the inclusion map. Then, if ρ is a smooth representation of G_F, the composition $\rho \circ \iota$ defines a representation of $\mathcal{W}_F$. With this, a representation τ of $\mathcal{W}_F$ is of Galois type if $\tau \cong \rho \circ \iota$ for some representation ρ of G_F. We first work towards giving a characterization of irreducible Weil representations of Galois type.

Lemma 4.1.1. *Let (ρ, V) be a an irreducible representation of $\mathcal{W}_F$. Then (ρ, V) is finite dimensional.*

Proof. See [BH06, §28.6] □

As before for representations of $\mathrm{GL}_2(F)$, we will write $\det\rho$ the character given by $x \mapsto \det\rho(x)$ for a given smooth representation ρ of $\mathcal{W}_F$ or of G_F.

Proposition 4.1.2. *Let (ρ, V) be an irreducible smooth representation of $\mathcal{W}_F$. Then the following are equivalent:*

(i) The group $\rho(\mathcal{W}_F)$ is of finite order.

(ii) The representation ρ is of Galois type.

(iii) The character $\det\rho$ has finite order.

For any irreducible representation ρ of $\mathcal{W}_F$, there exists an unramified character χ of $\mathcal{W}_F$ (a character which is trivial on I_F) such that $\chi \otimes \rho$ satisfies the equivalent conditions above.

Proof. Suppose that $\rho(\mathcal{W}_F)$ is finite and let $\Phi \in \mathcal{W}_F$ be a Frobenius element. Then $\rho(\Phi)$ has order $d < \infty$. Any element $\omega \in G_F$ may be written $\omega = \sigma\Phi^a$ for some $\sigma \in I_F$ and $a \in \widehat{\mathbb{Z}}$. For such an a, there exists $\bar{a} \in \mathbb{Z}$ such that $\bar{a} \equiv a \pmod{d\widehat{\mathbb{Z}}}$. Moreover, this $\bar{a}$ is uniquely determined modulo $d\mathbb{Z}$. We may define a G_F-representation τ by $\tau(\sigma\Phi^a) = \rho(\sigma)\rho(\Phi)^{\bar{a}}$. Then τ is irreducible and smooth, and we have that $\tau \circ \iota \cong \rho$.

The fact that (ii) implies both (i) and (iii) is immediate, as a Galois representation will have both these properties by a standard compactness argument. Thus we assume (iii) and show (i). Again let $\Phi \in \mathcal{W}_F$ be a Frobenius element. Then, as $\rho(I_F)$ is finite (again, compactness), there is an integer $k \geq 1$ such that $\rho(\Phi^k) = \rho(\Phi)^k$ commutes with $\rho(I_F)$. On the other hand, $\rho(\Phi)^k$ vacuously commutes with the Frobenius part of $\mathcal{W}_F$. Thus, $\rho(\Phi)^k$ commutes with the entirety of $\mathcal{W}_F$ and is hence a scalar. Now if $\det\rho$ has finite order, then so too does the scalar $\rho(\Phi)^k$, whence $\rho(\mathcal{W}_F)$ is finite.

No matter what, the arguments of the previous paragraph show that if ρ is an irreducible smooth representation of $\mathcal{W}_F$, then there will be some integer $k \geq 1$ such that $\rho(\Phi)^k$ is a scalar c. Choose an unramified character χ of $\mathcal{W}_F$ such that $\chi(\Phi)^k = c$. Then $\chi^{-1} \otimes \rho$ satisfies the equivalent conditions of the proposition. □

An observation one may make upon reflecting on the previous two proofs is that, with respect to the representation theory, in the decomposition $\mathcal{W}_F \cong \langle\Phi\rangle \ltimes I_F$, the profinite inertia subgroup part is well-behaved, while the Frobenius part introduces complications. This intuition is further corroborated when analyzing the semisimplicity of a Weil representation.

Proposition 4.1.3. *Let (ρ, V) be a smooth, finite dimensional representation of $\mathcal{W}_F$ and let $\Phi \in \mathcal{W}_F$ be a Frobenius element. Then the following are equivalent.*

(i) The representation ρ is semisimple.

(ii) The automorphism $\rho(\Phi) \in \mathrm{GL}(V)$ is semisimple.

(iii) The automorphism $\rho(\Psi) \in \mathrm{GL}(V)$ is semisimple for any $\Psi \in \mathcal{W}_F$.

Proof. Obviously (iii) implies (ii), so suppose (ii). As we have seen before, there is some integer $d \geq 1$ such that $\rho(\Phi)^d$ commutes with the finite group $\rho(I_F)$. Since $\rho(\Phi)$ is semisimple, so too must be $\rho(\Phi)^d$. Therefore, ρ restricted to the open subgroup $\langle I_F, \Phi^d\rangle$ is semisimple. This is equivalent to ρ being semisimple (cf. [BH06, §2.7]). Thus (ii) implies (i).

Now suppose (i) holds. Let $\Psi \in \mathcal{W}_F$. If $\Psi \in I_F$ then the result is immediate, so we accordingly assume $\Psi \notin I_F$. Therefore $\Psi \in \mathcal{W}_E$ for some finite extension E/F and $\mathcal{W}_E = \langle I_E, \Psi\rangle$. By assumption, $\rho|_{\mathcal{W}_E}$ is semisimple, and so we may as well assume that $F = E$. Moreover, since ρ is semisimple and thus a sum of irreducible representations, we can assume ρ is irreducible, and the result will hold for the sum of such irreducibles. From the proof of the previous proposition, we see that $\rho(\Psi) = c\varphi$, where $c \in \mathbb{C}^\times$ and φ is an operator of finite order. Therefore $\rho(\Psi)$ is semisimple as desired. □

We use the following notation moving forward. Let E/F be a finite extension in F^{sep}, so that $\mathcal{W}_E \subseteq \mathcal{W}_F$. Then, we abbreviate the induction $\mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}$ functor to $\mathrm{Ind}_{E/F}$. Similarly, the restriction functor $\mathrm{Res}_{\mathcal{W}_E}^{\mathcal{W}_F}$ will be written $\mathrm{Res}_{E/F}$. For the set of isomorphism classes semisimple n-dimensional smooth representations of $\mathcal{W}_F$, we put $\mathcal{G}_n^{\mathrm{ss}}(F)$. For the set of isomorphism classes of n-dimensional irreducible smooth representations of $\mathcal{W}_F$, we put $\mathcal{G}_n^0(F)$. With this, we have

$$\mathrm{Ind}_{E/F} \colon \mathcal{G}_n^{\mathrm{ss}}(E) \to \mathcal{G}_{nd}^{\mathrm{ss}}(F),$$

where $d = [E : F]$, and

$$\mathrm{Res}_{E/F} \colon \mathcal{G}_n^{\mathrm{ss}}(F) \to \mathcal{G}_n^{\mathrm{ss}}(E).$$

4.1.2 L-Functions Attached to Weil Representations

Let

$$\mathrm{art}_F \colon F^\times \xrightarrow{\sim} \mathcal{W}_F^{\mathrm{ab}}$$

be the local reciprocity map of local class field theory. This isomorphism allows one to identify characters of $F^\times$ with characters of $\mathcal{W}_F$, as any such character must factor through the Abelianization $\mathcal{W}_F^{\mathrm{ab}}$. It is known that to a character χ of $F^\times$, one may attach the L-function

$$L(\chi, s) = \begin{cases} (1 - \chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{otherwise.} \end{cases}$$

Fix a Haar measure dx on F, and let $d^*x = \|x\|^{-1}dx$. Then d^*x is a Haar measure on $F^\times$. For any compactly supported smooth function $f \in C_c^\infty(F^\times)$, we define the *local zeta function*

$$Z(f, \chi, s) = \int_{F^\times} f(x)\chi(x)\|x\|^s d^*x.$$

Working relative to a nontrivial additive character ψ of F, we have the *Fourier transform* of f,

$$\hat{f}(y) = \int_F f(x)\psi(xy)dx.$$

In his doctoral thesis ([Tat67]), Tate proved that these local zeta functions satisfy the functional equation

$$\frac{Z(\hat{f}, \check{\chi}, 1-s)}{L(\check{\chi}, 1-s)} = \varepsilon(\chi, s, \psi)\frac{Z(f, \chi, s)}{L(\chi, s)},$$

for some $\varepsilon(\chi, s, \psi) \in \mathbb{C}(q^{-s})$ which is independent of f. The $\varepsilon(\chi, s, \psi)$ are called *local epsilon factors*, and they are given by relatively simple expressions, which may be found in [Tat79, §3].

If χ' is a character of $\mathcal{W}_F$, then $\chi' \circ \mathrm{art}_F$ is a character of $F^\times$. Using this identification, we define L-functions and local epsilon factors of characters of the Weil group. We set

$$L(\chi', s) = L(\chi' \circ \mathrm{art}_F, s),$$

and

$$\varepsilon(\chi', s, \psi) = \varepsilon(\chi' \circ \mathrm{art}_F, s, \psi).$$

We extend these definitions to semisimple smooth Weil representations. The L-functions are relatively strightforward to define. First we define them for an irreducible smooth representation $\rho \in \mathcal{G}_n^{\mathrm{ss}}(F)$. If $n = 1$, then ρ is a character and the definitions are as above. If $n \geq 2$, we set

$$L(\rho, s) = 1.$$

Using these, we extend to all semisimple representations by

$$L(\rho_1 \oplus \rho_2, s) = L(\rho_1, s)L(\rho_2, s).$$

The analogue of the local epsilon factor for semisimple Weil representations requires more effort to define. We first define, for a finite separable extension E/F the *regular representation*,

$$R_{E/F} = \mathrm{Ind}_{E/F}\, \mathbb{1}_E,$$

where $\mathbb{1}_E$ is the trivial character on $\mathcal{W}_E$. This representation has determinant

$$\varkappa_{E/F} = \det R_{E/F},$$

which is of order 2. We continue with the notation $\psi_E = \psi \circ \mathrm{tr}_E$ for a character $\psi \in \widehat{F}$ and we write $\mathcal{G}^{\mathrm{ss}}(F) = \bigcup_{n \geq 1} \mathcal{G}_n^{\mathrm{ss}}(F)$.

Theorem 4.1.4. *Let ψ be a nontrivial additive character of F. Let E/F range over all finite extensions of F contained in F^{sep}. Then there is a unique family of functions*

$$\begin{aligned} \mathcal{G}^{\mathrm{ss}}(E) &\to \mathbb{C}[q^s, q^{-s}]^\times \\ \rho &\mapsto \varepsilon(\rho, s, \psi_E) \end{aligned}$$

with the following properties:

(i) *If* χ *is a character of* $E^\times$, *then*

$$\varepsilon(\chi, s, \psi_E) = \varepsilon(\chi \circ \mathrm{art}_E{}^{-1}, s, \psi_E).$$

(ii) *For* $\rho_1, \rho_2 \in \mathcal{G}^{\mathrm{ss}}(E)$ *one has*

$$\varepsilon(\rho_1 \oplus \rho_2, s, \psi_E) = \varepsilon(\rho_1, s, \psi_E)\varepsilon(\rho_2, s, \psi_E).$$

(iii) *If* $E \supseteq K \supseteq F$ *and* $\rho \in \mathcal{G}_n^{\mathrm{ss}}(E)$, *then*

$$\frac{\varepsilon(\mathrm{Ind}_{E/K}\,\rho, s, \psi_K)}{\varepsilon(\rho, s, \psi_E)} = \frac{\varepsilon(R_{E/K}, s, \psi_K)^n}{\varepsilon(\mathbb{1}_E, s, \psi_E)^n}.$$

For $\rho \in \mathcal{G}^{\mathrm{ss}}(F)$, the $\varepsilon(\rho, s, \psi)$ in the theorem is called the *Langlands-Deligne local epsilon factor* of ρ (relative to ψ). The proof of the existence and uniqueness of such a quantity satisfying the desiderata of the theorem is quite involved and we do not include it here. One may find a proof in [Del73] or [BH06, §30]. We collect some of the properties which the Langlands-Deligne local epsilon factors enjoy in the following proposition, whose proof is also found in [BH06, §30].

Proposition 4.1.5. *Let* ψ *be a nontrivial additive character of* F *and let* $\rho \in \mathcal{G}^{\mathrm{ss}}(F)$. *Then,*

(i) *There is an integer* $n(\rho, \psi)$ *such that*

$$\varepsilon(\rho, s, \psi) = q^{n(\rho,\psi)(\frac{1}{2}-s)}\varepsilon(\rho, 1/2, \psi).$$

(ii) *Let* $a \in F^\times$. *Then,*

$$\varepsilon(\rho, s, a\psi) = \det\rho(a)\|a\|^{\dim(\rho)(s-\frac{1}{2})}\varepsilon(\rho, s, \psi),$$

and

$$n(\rho, a\psi) = n(\rho, \psi) + v_F(a)\dim\rho.$$

(iii) *The Langlands-Deligne local epsilon factors satisfy the functional equation*

$$\varepsilon(\rho, s, \psi)\varepsilon(\check{\rho}, 1-s, \psi) = \det\rho(-1).$$

(iv) *There exists* $n_\rho \in \mathbb{Z}$ *such that if* χ *is a character of* $F^\times$ *of level* $k \geq n_\rho$, *then*

$$\varepsilon(\chi \otimes \rho, s, \psi) = \det\rho(c(\chi))^{-1}\varepsilon(\chi, s, \psi)^{\dim\rho}$$

for any $c(\chi) \in F^\times$ *such that* $\chi(1+x) = \psi(c(\chi)x)$ *for all* $x \in \mathfrak{p}^{[k/2]+1}$.

4.2 Weil-Deligne Representations

For an element $x \in \mathcal{W}_F$, we put $\|x\| = q^{-v_F(x)}$ where $v_F\colon \mathcal{W}_F \to \mathbb{Z}$ is the canonical map sending geometric Frobenius to 1.

Definition 4.2.1. A *Weil-Deligne Representation* of $\mathcal{W}_F$ is a triple $(\rho, V, \mathfrak{n})$ where (ρ, V) is a smooth representation of $\mathcal{W}_F$ of finite dimension, and $\mathfrak{n} \in \mathrm{End}_{\mathbb{C}}(V)$ is a nilpotent linear map satisfying

$$\rho(x)\mathfrak{n}\rho(x)^{-1} = \|x\|\mathfrak{n}$$

for all $x \in \mathcal{W}_F$. The Weil-Deligne representation $(\rho, V, \mathfrak{n})$ is said to be *semisimple* if the underlying sooth representation (ρ, V) is semisimple. We put $\mathcal{G}_n(F)$ for the set of semisimple n-dimensional Weil-Deligne representations, up to equivalence.

The definition of a Weil-Deligne representation is best motivated by considering ℓ-adic representations and seeing how the structure of a Weil-Deligne representation naturally arises.

4.2.1 ℓ-Adic Representations

Fix a prime $\ell \neq p$. By an *ℓ-adic representation* of $\mathcal{W}_F$, we mean a representation (ρ, V) of $\mathcal{W}_F$ where V is a $\overline{\mathbb{Q}_\ell}$-vector space, where $\overline{\mathbb{Q}_\ell}$ is an algebraic closure of $\mathbb{Q}_\ell$. In this setting, the notion of a smooth representation holds just the same as when we worked over $\mathbb{C}$. Moreover, if V is a d-dimensional $\overline{\mathbb{Q}_\ell}$-vector space, then the choice of a basis identifies $\mathrm{GL}(V) \cong \mathrm{GL}_d(\overline{\mathbb{Q}_\ell})$, whence $\mathrm{GL}(V)$ inherits a topology which is independent of the choice of basis. Thus one may discuss *continuous ℓ-adic representations*, as those (ρ, V) where $\rho\colon \mathcal{W}_F \to \mathrm{GL}(V)$ is continuous. Every smooth representation is continuous, but the converse need not hold.

Let V be a finite dimensional $\overline{\mathbb{Q}_\ell}$-vector space. If $\mathfrak{n} \in \mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V)$ is nilpotent, then

$$\exp \mathfrak{n} = \sum_{j=0}^{\infty} \frac{\mathfrak{n}^j}{j!}$$

is a finite sum and defines a unipotent element of $\mathrm{GL}(V)$. Conversely, if $\mathfrak{u} \in \mathrm{GL}(V)$ is unipotent, so that $\mathfrak{u} = 1 + \mathfrak{n}$ for some nilpotnent $\mathfrak{n}$, then

$$\log \mathfrak{u} = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{\mathfrak{n}^j}{j}$$

is a nilpotent element in $\mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V)$.

Remark 12. Let C be any field of characteristic zero and V a finite dimensional C-vector space. Then the series defined above define unipotent and nilpotent elements, respectively, if $\mathfrak{n} \in \mathrm{End}_C(V)$ and $\mathfrak{u} \in \mathrm{GL}(V)$ are nilpotent and unipotent elements. That is, there is nothing special about $\overline{\mathbb{Q}_\ell}$ in the constructions.

We put $P_F = \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{tr}})$, where F^{tr} is the maximal tamely ramified extension of F, for the wild inertia subgroup. The wild inertia subgroup fits into the short exact sequence

$$1 \to P_F \to I_F \to \prod_{\ell \neq p} \mathbb{Z}_\ell \to 0,$$

as $I_F/P_F \cong \mathrm{Gal}(F^{\mathrm{tr}}/F^{\mathrm{nr}}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$. The isomorphism $f\colon I_F/P_F \to \prod_{\ell \neq p} \mathbb{Z}_\ell$ is uniquely determined up to multiplication by $\prod_{\ell \neq p} \mathbb{Z}_\ell^\times$. Moreover, the group $\mathrm{Gal}(F^{\mathrm{nr}}/F)$ acts on the quotient I_F/P_F by conjugation. Upon computing how a Frobenius element $\Phi \in \mathrm{Gal}(F^{\mathrm{nr}}/F)$ acts on the generators of $F^{\mathrm{tr}}/F^{\mathrm{nr}}$ (which are nth roots of a uniformizer ϖ of

F for n not dividing the residual characteristic p), one sees that the action is transported to $\prod_{\ell \neq p} \mathbb{Z}_\ell$ as

$$f(\Phi\sigma\Phi^{-1}) = q^{-1} f(\sigma).$$

It follows that there is a continuous surjection $t\colon I_F \to \mathbb{Z}_\ell$, unique up to multiplication by $\mathbb{Z}_\ell^\times$, whose kernel lies in the exact sequence

$$1 \to P_F \to \ker t \to \prod_{m \neq \ell, p} \mathbb{Z}_m \to 0.$$

Similarly, one has

$$t(gxg^{-1}) = \|g\| t(x), \tag{4.1}$$

for $x \in I_F$ and $g \in \mathcal{W}_F$.

The following theorem is due to Grothendieck.

Theorem 4.2.2 (Grothendieck's ℓ-Adic Monodromy Theorem)**.** *Let (σ, V) be a finite dimensional continuous representation of $\mathcal{W}_F$ over $\overline{\mathbb{Q}_\ell}$. Then there exists a unique nilpotent $\mathfrak{n}_\sigma \in \mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V)$ such that*

$$\sigma(x) = \exp(t(x)\mathfrak{n}_\sigma)$$

for all x in some open subgroup of I_F.

Grothendieck's ℓ-Adic Monodromy Theorem is proven, with some more generality, in the appendix of [ST68].

Let us observe the consequences of the ℓ-Adic Monodromy Theorem. Suppose that (σ, V) is a continuous finite dimensional representation of $\mathcal{W}_F$ over $\overline{\mathbb{Q}_\ell}$, as in the theorem. Then, there exists an open subgroup $U \subseteq I_F$ such that for all $x \in U$, one has

$$\sigma(x) = \exp(t(x)\mathfrak{n}_\sigma)$$

for some unique nilpotent operator $\mathfrak{n}_\sigma$. Shrinking if necessary, we may assume U is normal in $\mathcal{W}_F$, so that $gxg^{-1} \in U$ for all $g \in \mathcal{W}_F$. Then, we get

$$\sigma(g)\exp(t(x)\mathfrak{n}_\sigma)\sigma(g)^{-1} = \exp(t(gxg^{-1})\mathfrak{n}_\sigma). \tag{4.2}$$

Using (4.1), the right hand side of the above becomes

$$\exp(\|g\| t(x)\mathfrak{n}_\sigma).$$

Then, upon computing the exponential series on both sides of (4.2), which, by the nilpotency of $\mathfrak{n}_\sigma$ have finitely many terms, the unicity of $\mathfrak{n}_\sigma$ implies

$$\sigma(g)\mathfrak{n}_\sigma\sigma(g)^{-1} = \|g\|\mathfrak{n}_\sigma \tag{4.3}$$

for all $g \in \mathcal{W}_F$.

With this, we fix a Frobenius element $\Phi \in \mathcal{W}_F$, and define the map $\sigma_\Phi\colon \mathcal{W}_F \to \mathrm{GL}(V)$ by

$$\sigma_\Phi(\Phi^a x) = \sigma(\Phi^a x)\exp(-t(x)\mathfrak{n}_\sigma),$$

where $a \in \mathbb{Z}$ and $x \in I_F$. By Grothendieck's ℓ-Adic Monodromy Theorem, σ_Φ is trivial on an open subgroup of I_F and so defines a smooth ℓ-adic representation of $\mathcal{W}_F$. In particular σ_Φ is a continuous ℓ-adic Weil representation, and so by (4.3), the triple $(\sigma_\Phi, V, \mathfrak{n}_\sigma)$ is a Weil-Deligne representation.

Let $\mathrm{Rep}^{\mathrm{f}}_{\overline{\mathbb{Q}_\ell}}(\mathcal{W}_F)$ denote the category of finite dimensional continuous representations of $\mathcal{W}_F$ over $\overline{\mathbb{Q}_\ell}$ and let $\text{D-Rep}_{\overline{\mathbb{Q}_\ell}}(\mathcal{W}_F)$ denote the category of Weil-Deligne representations over $\overline{\mathbb{Q}_\ell}$. The constructions above suggest the assignment

$$(\sigma, V) \mapsto (\sigma_\Phi, V, \mathfrak{n}_\sigma)$$

is a functor

$$\mathrm{Rep}^{\mathrm{f}}_{\overline{\mathbb{Q}_\ell}}(\mathcal{W}_F) \to \text{D-Rep}_{\overline{\mathbb{Q}_\ell}}(\mathcal{W}_F).$$

The reality is even better.

Theorem 4.2.3. *Let $\Phi \in \mathcal{W}_F$ be a Frobenius element and let $t\colon I_F \to \mathbb{Z}_\ell$ be a continuous surjection. Then, the assignment $(\sigma, V) \mapsto (\sigma_\Phi, V, \mathfrak{n}_\sigma)$ is functorial and induces an equivalence of categories*

$$\mathrm{Rep}^{\mathrm{f}}_{\overline{\mathbb{Q}_\ell}}(\mathcal{W}_F) \cong \text{D-Rep}_{\overline{\mathbb{Q}_\ell}}(\mathcal{W}_F).$$

Moreover, the isomorphism class of the Weil-Deligne representation depends only on that of (σ, V). In particular, the functor is independent of choice of Φ and t.

Proof. See [BH06, §32.6] □

As the equivalence in theorem 4.2.3 is independent of the choices of a Frobenius element and surjection $t\colon I_F \to \mathbb{Z}_\ell$, the assignment $(\sigma, V) \mapsto (\sigma_\Phi, V, \mathfrak{n}_\sigma)$ gives a canonical bijection between the set of isomorphism classes of finite dimensional continuous representations of $\mathcal{W}_F$ over $\overline{\mathbb{Q}_\ell}$ and set of isomorphism classes of Weil-Deligne representations over $\overline{\mathbb{Q}_\ell}$. Any choice of isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ induces a bijection from either of the aformentioned sets to the set of isomorphism classes of Weil-Deligne representations over $\mathbb{C}$. We remark, however, that this bijection is certainly *not* canonical.

We say that a Weil-Deligne representation $(\rho, V, \mathfrak{n})$ is *semisimple* if the underlying smooth representation (ρ, V) is semisimple.

Definition 4.2.4. A finite dimensional continuous representation (σ, V) of $\mathcal{W}_F$ is Φ-*semisimple* (read "*Frobenius semisimple*") if there exists a Frobenius element $\Phi \in \mathcal{W}_F$ such that $\sigma(\Phi)$ is semisimple.

It turns out that Φ-semisimplicity of a finite dimensional continuous representation of $\mathcal{W}_F$ and semisimplicity of its associated Weil-Deligne representation are equivalent notions.

Proposition 4.2.5. *Let (σ, V) be a finite dimensional continuous representation of $\mathcal{W}_F$ over $\overline{\mathbb{Q}_\ell}$. The following are equivalent:*

(i) The representation (σ, V) is Φ-semisimple.

(ii) The associated Weil-Deligne representation $(\sigma_\Phi, V, \mathfrak{n}_\sigma)$ is semisimple.

(iii) The automorphism $\sigma(g)$ is semisimple for every $g \in \mathcal{W}_F \smallsetminus I_F$.

Proof. Recall that, by construction, for any Frobenius element $\Phi \in \mathcal{W}_F$, we have

$$\sigma_\Phi(\Phi) = \sigma(\Phi)\exp(-t(1)\mathfrak{n}_\sigma) = \sigma(\Phi).$$

Thus, the equivalence of (i) and (ii) follows immediately from proposition 4.1.3. Also, the fact that (iii) implies (i) is immediate. Thus it remains just to show (i) implies (iii).

Suppose that $g = \Phi x$ for some $x \in \mathcal{W}_F$ and Φ a Frobenius element. Then the operators $\sigma(g)$ and $\sigma(\Phi)$, which is assumed semisimple, are conjugate, whence $\sigma(g)$ is semisimple. A generic $g \in \mathcal{W}_F$ is of the form $\Phi^a x$, for $x \in I_F$ and $a \in \mathbb{Z}$. In this case, we restrict to $\mathcal{W}_E$ for E/F unramified of degree $|a|$. Then the same argument shows $\sigma(g)$ is semisimple. □

Combining this result with theorem 4.2.3, we arrive at the following.

Theorem 4.2.6. *Let $n \geq 1$ be an integer. Then the set of isomorphism classes of the following two types of representations are in canonical bijection:*

(i) n-dimensional semisimple Weil-Deligne representations over $\overline{\mathbb{Q}_\ell}$.

(ii) n-dimensional continuous Φ-semisimple representations of $\mathcal{W}_F$ over $\overline{\mathbb{Q}_\ell}$.

Moreover, a choice of isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ induces a bijection with either of the above two sets with the set of isomorphism classes of n-dimensional semisimple Weil-Deligne representations over $\mathbb{C}$.

Thus we see that by considering finite dimensional continuous representations of $\mathcal{W}_F$ (which appear in "nature", e.g. via étale cohomology) which are well-behaved with respect to the action of Frobenius (are Φ-semisimple), the structure of a semisimple Weil-Deligne representation arises. The connection is made clear by the ℓ-Adic Monodromy Theorem.

4.2.2 L-Functions and Local Epsilon Factors of Weil-Deligne Representations

We conclude this chapter by recording how L-functions and local epsilon factors are defined for Weil-Deligne representations. Let $(\rho, V, \mathfrak{n})$ be a semisimple Weil-Deligne representation (over $\mathbb{C}$, say). Let $V_\mathfrak{n} = \ker \mathfrak{n}$. The space $V_\mathfrak{n}$ carries a semisimple representation $\rho_\mathfrak{n}$ of $\mathcal{W}_F$: Let $v \in V_\mathfrak{n}$, so that $\mathfrak{n}v = 0$. Suppose that $V_\mathfrak{n}$ is not $\mathcal{W}_F$-stable, so that for some $g \in \mathcal{W}_F$ we have $\rho(g)v \notin V_\mathfrak{n}$. This implies then that

$$\rho(g)^{-1}\mathfrak{n}\rho(g)v \neq 0,$$

which is clearly a contradiction.

We accordingly set

$$L((\rho, V, \mathfrak{n}), s) = L(\rho_\mathfrak{n}, s).$$

The contragredient $(\rho, V, \mathfrak{n})^\vee$ of $(\rho, V, \mathfrak{n})$ is defined to be $(\check{\rho}, \check{V}, -\check{\mathfrak{n}})$, where $\check{\mathfrak{n}} \in \mathrm{End}_\mathbb{C}(\check{V})$ is the transpose of $\mathfrak{n}$. The local epsilon factor of the Weil-Deligne representation $(\rho, V, \mathfrak{n})$ is

$$\varepsilon((\rho, V, \mathfrak{n}), s, \psi) = \varepsilon(\rho, s, \psi)\frac{L(\check{\rho}, 1-s)L(\rho_\mathfrak{n}, s)}{L(\rho, s)L(\check{\rho}_\mathfrak{n}, 1-s)}.$$

Chapter 5

Automorphic L-Functions and the Godement-Jacquet Functional Equation

In the previous chapter we saw how, through the local reciprocity map, one may define L-functions and local epsilon factors corresponding to characters of the Weil group $\mathcal{W}_F$. From there, we were able to extend the definitions to higher dimensional Weil representations and Weil-Deligne representations. Ultimately, the Langlands correspondence will demand some sort of compatibility between the L-functions and local epsilon factors of Weil-Deligne representations and analogous L-functions and local epsilon factors associated to representations of $\mathrm{GL}_2(F)$. In this chapter we define these L-functions and epsilon factors as well as the functional equation which they satisfy, called the *Godement-Jacquet functional equation*. The chapter culminates in the so-called "Converse Theorem," which states that the L-functions and epsilon factors associated to a irreducible smooth representation (π, V) of $G = \mathrm{GL}_2(F)$ essentially determines π. Here we largely just state and collect the results regarding these L-functions which will be essential to showing the local Langlands correspondence, directing the reader to [BH06, Ch. 6] for a complete treatment of the theory.

5.1 The Functional Equation

Let $A = \mathrm{Mat}_2(F)$ and put $\mathfrak{M} = \mathrm{Mat}_2(\mathcal{O})$. Then, the space of compactly supported locally constant functions on A, $C_c^\infty(A)$, is spanned by the characteristic functions of cosets $a + \mathfrak{p}^j\mathfrak{M}$, for $a \in A$ and $j \in \mathbb{Z}$. We work relative to a fixed nontrivial additive character $\psi \in \widehat{F}$ and form the additive character ψ_A of A by defining $\psi_A = \psi \circ \mathrm{tr}_A$.

Let $\Phi \in C_c^\infty(A)$. We define its Fourier transform, $\hat{\Phi}$, by

$$\hat{\Phi} = \int_G \Phi(x)\psi_A(xy)d\mu(y),$$

relative to a Haar measure μ on A. The theory of Fourier analysis on locally compact groups ensures that $\hat{\Phi} \in C_c^\infty(A)$ as well, and moreover that there exists a unique *self-dual Haar measure (with respect to ψ)* μ_ψ^A on A for which the Fourier inversion formula holds:

$$\hat{\hat{\Phi}}(x) = \Phi(-x),$$

for $\Phi \in C_c^\infty(A)$.

For (π, V) an irreducible smooth representation of G, we denote by $\mathcal{C}(\pi)$ be the space of coefficients of π, as in section 3.1. For $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$, we consider zeta integrals of the form

$$\zeta(\Phi, f, s) = \int_G \Phi(x) f(x) \|\det x\|^s d^*x,$$

where d^*x is shorthand for $d\mu^*(x)$ and μ^* is a Haar measure on G. With this we may define the L-function associated to π.

Theorem 5.1.1 (Theorem-Definition)**.** *Let (π, V) be an irreducible smooth representation of G and let $\zeta(\Phi, f, s)$ be a zeta integral as above. Then:*

(i) There exists $s_0 \in \mathbb{R}$ such that $\zeta(\Phi, f, s)$ is absolutely and uniformly convergent in the half-plane $\mathrm{Re}\, s > s_0$. Moreover the integral represents a rational function in q^{-s}.

(ii) Let $Z(\pi) = \{\zeta(\Phi, f, s + 1/2) \mid \Phi \in C_c^\infty(A), f \in \mathcal{C}(\pi)\}$. Then, there exists a unique polynomial $P_\pi(X) \in \mathbb{C}[X]$ such that $P_\pi(0) = 1$ and

$$Z(\pi) = P_\pi(q^{-s})^{-1}\mathbb{C}[q^s, q^{-s}].$$

We then define $L(\pi, s) = P_\pi(q^{-s})^{-1}$.

Let (π, V) be an irreducible smooth representation and let $f \in \mathcal{C}(\pi)$. Then, we define $\check{f} \in \mathcal{C}(\check{\pi})$ by

$$\check{f} \colon g \mapsto f(g^{-1}).$$

The process $f \mapsto \check{f}$ defines an isomorphism $\mathcal{C}(\pi) \cong \mathcal{C}(\check{\pi})$. The main result is the following.

Theorem 5.1.2. *Let (π, V) be an irreducible smooth representation of G. Then, there exists a unique rational function $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$\zeta(\hat{\Phi}, \check{f}, 3/2 - s) = \gamma(\pi, s, \psi)\zeta(\Phi, f, 1/2 + s)$$

for all $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$.

The equation appearing in theorem 5.1.2 is called the *Godement-Jacquet functional equation.* We then define the *Godement-Jacquet local epsilon factor*

$$\varepsilon(\pi, s, \psi) = \gamma(\pi, s, \psi)\frac{L(\pi, s)}{L(\check{\pi}, 1 - s)}.$$

Proposition 5.1.3. *The Godement-Jacquet local epsilon factor satisfies the functional equation*

$$\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1 - s, \psi) = \omega_\pi(-1).$$

Moreover, we have

$$\varepsilon(\pi, s, \psi) = aq^{bs}$$

for some $a \in \mathbb{C}^\times$ and $b \in \mathbb{Z}$.

In light of the proposition above, we may write

$$\varepsilon(\pi, s, \psi) = q^{n(\pi,\psi)(\frac{1}{2} - s)}\varepsilon(\pi, 1/2, \psi)$$

for some $n(\pi, \psi) \in \mathbb{Z}$.

We now state some results to assist in the computations of L-functions and local epsilon factors. These will be delineated by whether the representation (π, V) is supercuspidal or not. Firstly, in the supercuspidal case:

Proposition 5.1.4. *For $c \in \mathrm{GL}_2(F)$, let ψ_c be the character defined by $\psi_c(x) = \psi_A(c(x-1))$. Let $(\mathfrak{A}, J, \Lambda)$ be a cuspidal type in G and let n be the least integer such that $U_{\mathfrak{A}}^{n+1} \subseteq \ker \Lambda$. Choose $c \in J$ such that $\Lambda|_{U_{\mathfrak{A}}^{n+1}}$ is a multiple of ψ_c. If $\pi = \text{c-Ind}_J^G \Lambda$, then*

$$\varepsilon(\pi, 1/2, \psi) = q^a \sum_x \operatorname{tr} \Lambda^\vee(cx)\psi_A(cx),$$

where x ranges over $U_{\mathfrak{A}}^{[(n+1)/2]}/U_{\mathfrak{A}}^{[n/2]+1}$ and, writing $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$,

$$q^a = \begin{cases} (\dim \Lambda)^{-1} & \text{if } n \text{ is odd,} \\ (\dim \Lambda)^{-1}(\mathfrak{A} : \mathfrak{P})^{-1/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. See [BH06, §25.5] □

The main result for representations in the principal series is the following. Recall we put $\boldsymbol{\iota}_B^G$ for the normalized induction functor from B to G (the subgroups B and T of G are as in Chapter 2).

Theorem 5.1.5. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and let π be a G-composition factor of $\boldsymbol{\iota}_B^G \chi$. For any nontrivial additive character ψ of F, we have*

$$L(\pi, s) = L(\chi_1, s)L(\chi_2, s)$$
$$\varepsilon(\pi, s, \psi) = \varepsilon(\chi_1, s, \psi)\varepsilon(\chi_2, s, \psi),$$

except when $\pi \cong \varphi \cdot \mathrm{St}_G$ for an unramified character φ of $F^\times$. In this case we have

$$L(\pi, s) = L(\varphi, s + 1/2)$$

and

$$\varepsilon(\pi, s, \psi) = -\varepsilon(\varphi, s, \psi).$$

We conclude this short chapter on the L-functions of representations of $\mathrm{GL}_2(F)$ by recording the *Converse Theorem.* Recall that for a character χ of $F^\times$ and an irreducible smooth representation π of G, we denote the twist of π by χ as $\chi\pi$, which is the representation $x \mapsto \chi(\det x)\pi(x)$.

Theorem 5.1.6 (Converse Theorem)**.** *Let ψ be a nontrivial additive character of F and let π_1, π_2 be two irreducible smooth representations of G. If*

$$L(\chi\pi_1, s) = L(\chi\pi_2, s)$$

and

$$\varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi)$$

for all characters χ of $F^\times$, then $\pi_1 \cong \pi_2$.

Thus we see that the L-functions and Godement-Jacquet local epsilon factors carry an immense amount of information about their associated representations.

Chapter 6

The Local Langlands Correspondence for $\mathrm{GL}_2(F)$

We now have developed all the necessary machinery to state and prove the local Langlands correspondence for $G = \mathrm{GL}_2(F)$. The local Langlands correspondence is a canonical bijection between the set of equivalence classes of irreducible smooth representations of G and 2-dimensional semisimple Weil-Deligne representations of $\mathcal{W}_F$. Of course, between any two sets of the same cardinality we have a plethora of bijections. The canonicity of the local Langlands correspondence is realized in compatibility between the L-functions and epsilon factors on both sides, as well as the bijection commuting with twisting by characters of $F^\times$.

The correspondence on the automorphic ($\mathrm{GL}_2(F)$) side will, perhaps unsurprisingly, be delineated between those representations in the principal series and the supercuspidal representations. On the Galois side, this delineation manifests between the non-irreducible and irreducible Weil-Deligne representations, respectively. The bijection between principal series representations of G and non-irreducible semisimple Weil-Deligne representations is relatively quick to state and prove. Indeed, we will just construct a map and show it satisfies the desiderata. On the other hand, the construction of the bijection between supercuspidal representations of G and irreducible Weil-Deligne representations requires considerably more care.

We remark, one final time, that the bijection as described in this chapter is a bijection only when the residual characteristic of F is $p \neq 2$. This is a reflection of the same limitation of the Tame Parametrization Theorem (theorem 3.7.5).

We will use the notation $\mathscr{A}_2(F)$ for the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_2(F)$. Similarly, we put $\mathcal{G}_2(F)$ for the set of 2-dimensional semisimple Weil-Deligne representations of $\mathcal{W}_F$. If χ is a character of $\mathcal{W}_F$, we also write χ for the character $\chi \circ \mathrm{art}_F$ of $F^\times$. With this, we may formally state the local Langlands correspondence.

Theorem 6.0.1 (Local Langlands Correspondence)**.** *Let ψ be a nontrivial additive character of F. Then, there exists a unique bijection*

$$\mathbf{LLC}\colon \mathcal{G}_2(F) \to \mathscr{A}_2(F)$$

such that

$$L(\chi \mathbf{LLC}(\rho), s) = L(\chi \otimes \rho, s), \tag{6.1}$$

$$\varepsilon(\chi \mathbf{LLC}(\rho), s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi) \tag{6.2}$$

for all $\rho \in \mathcal{G}_2(F)$ and characters χ of $F^\times$. The bijection and relations above hold for all nontrivial additive characters of F.

First, one must remark that the unicity of the map **LLC** as described in the theorem is immediate by the Converse Theorem (theorem 5.1.6).

As eluded to in the preamble, the proof of the local Langlands correspondence will be split into two cases. Indeed, we have a decomposition

$$\mathcal{G}_2(F) = \mathcal{G}_2^0(F) \cup \mathcal{G}_2^1(F),$$

where $\mathcal{G}_2^0(F)$ is the set of equivalence classes of irreducible Weil-Deligne representations and $\mathcal{G}_2^1(F)$ is that of non-irreducible semisimple Weil-Deligne representations (more precisely, this means those triples $(\rho, V, \mathfrak{n})$ where the underlying (ρ, V) is a reducible representation). We observe that, for $(\rho, V, \mathfrak{n}) \in \mathcal{G}_2^0(F)$, since $V_\mathfrak{n} = \ker \mathfrak{n}$ is a W_F-stable subspace of V and a nilpotent operator will always have a nonzero kernel, it is forced that $\mathfrak{n} = 0$ by the irreducibility of ρ. Therefore we may as well consider $\mathcal{G}_2^0(F)$ to be the set of isomorphism classes of irreducible smooth representations (ρ, V) of $\mathcal{W}_F$.

On the $\mathrm{GL}_2(F)$ side, the set $\mathscr{A}_2(F)$ admits a decomposition

$$\mathscr{A}_2(F) = \mathscr{A}_2^0(F) \cup \mathscr{A}_2^1(F)$$

where $\mathscr{A}_2^0(F)$ is the set of equivalence classes of supercuspidal representations of G and $\mathscr{A}_2^1(F)$ consists of equivalence classes of irreducible smooth representations of G in the principal series. We first show that the correspondence **LLC** respects these decompositions.

Proposition 6.0.2. *Let $\pi \in \mathscr{A}_2(F)$. Then $\pi \in \mathscr{A}_2^0(F)$ if and only if $L(\varphi\pi, s) = 1$ for all characters φ of $F^\times$.*

Proof. The necessity is the content of [BH06, §24.5, Cor.] after observing that if π is supercuspidal then so too is $\varphi\pi$ for any character φ of $F^\times$. On the other hand, suppose that π is not supercuspidal. Then, π is a composition factor of $\boldsymbol{\iota}_B^G\chi$ for some character $\chi = \chi_1 \otimes \chi_2$ of $F^\times$. This means $\varphi\pi$ is a composition factor of $\boldsymbol{\iota}_B^G\varphi\chi$ where we recall $\varphi\chi = \varphi\chi_1 \otimes \varphi\chi_2$. Take $\varphi = \chi_2^{-1}$. Then it follows from theorem 5.1.5 that $L(\varphi\pi, s) \neq 1$. □

Similarly, on the Galois side we have:

Proposition 6.0.3. *Let $\rho \in \mathcal{G}_2(F)$. Then $\rho \in \mathcal{G}_2^0(F)$ if and only if $L(\varphi \otimes \rho, 1) = 1$ for all characters φ of $F^\times$.*

Proof. This follows directly from the definitions of L-functions for Weil-Deligne representations, as outlined in section 4.2.2. □

Since the map **LLC** should satisfy

$$L(\chi\mathbf{LLC}(\rho), s) = L(\chi \otimes \rho, s)$$

for all characters χ of $F^\times$, we conclude as a consequence of the previous two propositions that **LLC** must send $\mathcal{G}_2^0(F)$ to $\mathscr{A}_2^0(F)$ and $\mathcal{G}_2^1(F)$ to $\mathscr{A}_2^1(F)$.

6.1 The Map LLC on $\mathcal{G}_2^1(F)$

We remarked at the beginning of the chapter that the correspondence between $\mathcal{G}_2^1(F)$ and $\mathscr{A}_2^1(F)$ is the easier of the two to show. Accordingly, we prove it first.

Theorem 6.1.1. *There is a unique bijective map*

$$\mathbf{LLC}^1 \colon \mathcal{G}_2^1(F) \to \mathscr{A}_2^1(F)$$

such that

$$L(\chi \mathbf{LLC}^1(\rho), s) = L(\chi \otimes \rho, s)$$

for all characters χ of $F^\times$. Moreover, we have

$$\mathbf{LLC}^1(\chi \otimes \rho) = \chi \mathbf{LLC}^1(\rho)$$

for all characters χ of $F^\times$ and

$$\varepsilon(\mathbf{LLC}^1(\rho), s, \psi) = \varepsilon(\rho, s, \psi)$$

for all nontrivial additive characters ψ of F and $\rho \in \mathcal{G}_2^1(F)$.

Proof. We define the map $\mathbf{LLC}^1$ as follows: Let $(\rho, V, \mathfrak{n}) \in \mathcal{G}_2^1(F)$. Since $(\rho, V, \mathfrak{n})$ is semisimple, we have $\rho = \chi_1 \oplus \chi_2$ for characters χ_1, χ_2 of $F^\times$. We then form the character $\chi = \chi_1 \otimes \chi_2$ of the torus $T \subseteq G$ and consider the normalized induction $\pi = \boldsymbol{\iota}_B^G \chi$. If π is irreducible, then $(\rho, V, \mathfrak{n})$ being a Weil-Deligne representation forces $\mathfrak{n} = 0$. Suppose that the nilpotent $\mathfrak{n}$ is given as $\mathfrak{n} = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$. Then we have $\rho(x)\mathfrak{n}\rho(x)^{-1}$ is given by

$$\begin{pmatrix} \chi_1(x) & 0 \\ 0 & \chi_2(x) \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi_1(x)^{-1} & 0 \\ 0 & \chi_2(x)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \chi_1\chi_2(x)^{-1}a \\ 0 & 0 \end{pmatrix}.$$

On the other hand, we have $\rho(x)\mathfrak{n}\rho(x)^{-1} = \|x\|\mathfrak{n}$. But from the Irreducibility Criterion of Chapter 2 (theorem 2.2.4), the normalized induction $\boldsymbol{\iota}_B^G \chi$ is irreducible if and only if $\chi_1\chi_2^{-1}$ is *not* the map $x \mapsto \|x\|^{\pm 1}$. Thus we have $\mathfrak{n} = 0$. We then set $\mathbf{LLC}^1(\rho) = \pi$, and the relations on L-functions and epsilon factors follow by definition. Moreover, we have $\mathbf{LLC}^1(\varphi \otimes \rho) = \varphi \mathbf{LLC}^1(\rho)$ for any character φ of $F^\times$ immediately as well.

Thus it remains just to handle the case of $\boldsymbol{\iota}_B^G \chi$, with $\chi = \chi_1 \otimes \chi_2$ as before, being reducible. Thus, by our classification of representations in the principal series, there exists a character φ of $F^\times$ such that $\chi_1(x) = \varphi(x)\|x\|^{-1/2}$ and $\chi_2(x) = \varphi(x)\|x\|^{1/2}$. With this it is easy to see that there are only two representatives $(\rho, V, \mathfrak{n}) \in \mathcal{G}_2^1(F)$ with $\rho = \chi_1 \oplus \chi_2$, distinguished by whether or not $\mathfrak{n} = 0$. We then set

$$\mathbf{LLC}^1(\rho, V, 0) = \varphi \circ \det$$

and

$$\mathbf{LLC}^1(\rho, V, \mathfrak{n}) = \varphi \cdot \mathrm{St}_G.$$

In the first case, $\rho_{\mathfrak{n}} = \rho$, and so the associated L-function is

$$L((\rho, V, 0), s) = L(\chi_1 \oplus \chi_2, s) = L(\chi_1, s)L(\chi_2, s).$$

Then, by theorem 5.1.5,

$$L(\mathbf{LLC}^1(\rho), s) = L(\chi_1, s)L(\chi_2, s)$$

as well. The case of $\mathfrak{n} \neq 0$ is similar. □

6.2 The Map LLC on $\mathcal{G}_2^0(F)$

We now shift our attention to demonstrating the local Langlands correspondence between irreducible smooth representations of $\mathcal{W}_F$ and supercuspidal representations of $\mathrm{GL}_2(F)$, where the bulk of the work is required. Observe by propositions 6.0.2 and 6.0.3, the relationship between L-functions as demanded by theorem 6.0.1 is a moot point. Accordingly, it is our goal to prove:

Theorem 6.2.1. *Let ψ be a nontrivial additive character of F. There is a unique bijective map*

$$\mathbf{LLC}^0 \colon \mathcal{G}_2^0(F) \to \mathscr{A}_2^0(F)$$

such that

$$\varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\chi \mathbf{LLC}^0(\rho), s, \psi),$$

for all $\rho \in \mathcal{G}_2^0(F)$, multiplicative characters χ of $F^\times$, and nontrivial additive characters ψ of F.

First a simplifying proposition.

Proposition 6.2.2. *Let $\mathbf{LLC}^0$ be a map satisfying the relationship with epsilon factors as in theorem 6.2.1, relative to a nontrivial additive character ψ of F. Then,*

(i) If $\rho \in \mathcal{G}_2^0(F)$ and $\pi = \mathbf{LLC}^0(\rho)$ then the central character of π is $\omega_\pi = \det \rho$.

(ii) The map $\mathbf{LLC}^0$ satisfies

$$\varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\chi \mathbf{LLC}^0(\rho), s, \psi)$$

for all nontrivial additive characters ψ of F.

Proof. See [BH06, §33.4] □

6.2.1 Admissible Pairs and Irreducible Smooth Representations of $\mathcal{W}_F$

In section 3.7, we saw that there is a bijection between the set of isomorphism classes of admissible pairs $\mathbb{P}_2(F)$ and irreducible supercuspidal representations $\mathscr{A}_2^0(F)$. We will take advantage of this to construct our bijection between $\mathcal{G}_2^0(F)$ and $\mathscr{A}_2^0(F)$. Recall that an admissible pair $(E/F, \xi) \in \mathbb{P}_2(F)$ consists of a tamely ramified (vacuous for us with residual characteristic of $p \neq 2$) quadratic extension E/F and a character ξ of $E^\times$, subject to a condition on factoring through the norm map. By local class field theory, we may view ξ as a character of $\mathcal{W}_E$, and define

$$\rho_\xi = \mathrm{Ind}_{E/F}\, \xi.$$

We first prove the following:

Theorem 6.2.3. *Let $(E/F, \xi)$ be an admissible pair. Then, the induced representation $\rho_\xi = \mathrm{Ind}_{E/F}\, \xi$ of $\mathcal{W}_F$ is irreducible. Moreover, when the residual characteristic of F is $p \neq 2$, then the assignment $(E/F, \xi) \mapsto \rho_\xi$ induces a bijection*

$$\mathbb{P}_2(F) \to \mathcal{G}_2^0(F).$$

Proof. Let $(E/F, \xi) \in \mathbb{P}_2(F)$ and let $\sigma \in \mathrm{Gal}(E/F)$ be the nontrivial element. Since ξ, by definition, does not factor through the norm map $N_{E/F}$, the characters ξ and $\xi \circ \sigma$ of $E^\times$ are distinct. Passing via local class field theory to $\mathcal{W}_E$, this means the characters ξ and ξ^σ (which is $x \mapsto \xi(\sigma x \sigma^{-1})$) of $\mathcal{W}_E$ are distinct.

Since $\mathcal{W}_E$ is a normal subgroup of $\mathcal{W}_F$ of index 2, the induced representation ρ_ξ is irreducible as a result of $\xi \neq \xi^\sigma$. Moreover the equivalence class of ρ_ξ depends only on that of $(E/F, \xi)$ by construction. We say that a representation $\rho \in \mathcal{G}_2^0(F)$ is *unramified* if there exists an unramified character χ of $\mathcal{W}_F$ such that $\rho \cong \chi \otimes \rho$. With this, we have the following lemma.

Lemma 6.2.4. *The representation ρ_ξ is unramified if and only if E/F is unramified.*

Proof of Lemma. We put $\varkappa = \varkappa_{E/F}$ for the character of $F^\times$ which is trivial on $N_{E/F}(E^\times)$ (passing our previous definition of $\varkappa_{E/F}$ through art_F). We then have

$$\varkappa \otimes \rho_\xi \cong \mathrm{Ind}_{E/F}(\varkappa_E \otimes \xi),$$

where $\varkappa_E = \varkappa \circ N_{E/F}$. Clearly by assumption $\varkappa_E = 1$, then. Thus we see if E/F is unramified, then $\varkappa$ is an unramified character whence one implication follows.

On the other hand, suppose ρ_ξ is unramified and let χ be the unramified character of $F^\times$ such that $\rho \cong \chi \otimes \rho$ (as usual identifying characters of $F^\times$ with those of $\mathcal{W}_F$). If $\sigma \in \mathrm{Gal}(E/F)$ is nontrivial then $\xi^\sigma/\xi = \chi \circ N_{E/F}$. It follows then that $\xi|_{U_E^1}$ factors through $N_{E/F}$ which by the definition of admissible pairs forces that E/F is unramified. □

Returning to the proof of the theorem, we now show that $(E/F, \xi) \mapsto \rho_\xi$ is injective. Let $(E_1/F, \xi_1), (E_2/F, \xi_2) \in \mathbb{P}_2(F)$ and suppose that $\rho_{\xi_1} \cong \rho_{\xi_2}$. In the case that $E_1/F \cong E_2/F$ we might as well take $E_1 = E_2 = E$. Then, it follows from Clifford's theorem on restricted representations to finite index subgroups that

$$\rho_{\xi_1}|_{\mathcal{W}_E} \cong \xi_1 \oplus \xi_1^\sigma,$$

where again σ is the nontrivial element of $\mathrm{Gal}(E/F)$. This means that $\xi_2 = \xi_1$ or $\xi_2 = \xi_1^\sigma$. In either case, the admissible pairs $(E/F, \xi_1)$ and $(E/F, \xi_2)$ are isomorphic.

Suppose then that E_1/F and E_2/F are not isomorphic. Then at least one of the E_i is totally ramified. Thus, if we put $L = E_1E_2$ for the compositum, we have $[L : F] = 4$ and that the maximal unramified subextension E/F in L/F is of degree 2. The representation $\rho_{\xi_1} \cong \rho_{\xi_2}$ is fixed by tensoring with the characters $\varkappa_{E_1/F}$ and $\varkappa_{E_2/F}$, whence it is also fixed by tensoring with $\varkappa_{E/F}$. But E/F is unramified and so $\varkappa_{E/F}$ is an unramified character. The lemma then yields $E_1 = E_2 = E$, but we assumed $E_1 \neq E_2$. Therefore $(E/F, \xi) \mapsto \rho_\xi$ is injective.

To prove surjectivity, we first consider the case that ρ is an unramified representation of $\mathcal{W}_F$. In this case, by considering how $\rho \cong \chi \otimes \rho$ (for some unramified χ) acts on a Frobenius element, one may show that ρ is induced from some character ξ of an unramified quadratic extension E/F, whence $\rho = \rho_\xi$ for an admissible pair $(E/F, \xi)$.

Now suppose that ρ is not an unramified representation of $\mathcal{W}_F$. As ρ is 2-dimensional, the restriction of ρ to the wild inertia subgroup P_F (which is a pro p-group) decomposes into a sum of characters. Thus there is a finite tamely ramified extension K/F such that $\rho|_{\mathcal{W}_K} = \theta \oplus \theta'$ for characters θ and θ'. Then, the $\mathcal{W}_F$-stabilizer of θ is $\mathcal{W}_L$, for some quadratic L/F with $L \subseteq K$.

From the natural representation of $\mathcal{W}_L$ on the θ-isotypic subspace of ρ we get a character ξ of $\mathcal{W}_L$ such that $\rho = \mathrm{Ind}_{L.F}\, \xi$. Passing via art_F to view ξ as a character of $L^\times$, we claim that the pair $(L/F, \xi)$ is an admissible pair. Let σ be the nontrivial element of $\mathrm{Gal}(L/F)$. Then we have that $\xi^\sigma|_{\mathcal{W}_K} = \theta'$, and so $\xi \neq \xi^\sigma$. It follows then that ξ does not factor through $N_{E/F}$. Since ρ is not unramified and $rho \cong \mathrm{Ind}_{L/F}\,\xi$, we have that L/F is totally ramified. This means if $\xi|_{U_L^1}$ factors through $N_{L/F}$, then the character $\xi^\sigma\xi^{-1}$ is trivial on $U_L = U_F U_L^1$, and is thus unramified. Thus $\xi^\sigma = (\chi \circ N_{L/F})\xi$ for some nontrivial unramified character χ of $F^\times$. But then $\rho \cong \chi \otimes \rho$, which would mean ρ is unramified, which is a contradiction. Thus the pair $(L/F, xi)$ is admissible, as desired.

In the case that $\theta = \theta'$, it follows quickly that ρ must be unramified yielding the same contradiction. □

Thus we have a bijective map $\mathbb{P}_2(F) \to \mathcal{G}_2^0(F)$ given by $(E/F, \xi) \mapsto \rho_\xi$. On the other hand, in section 3.7 we demonstrated the bijective map $\mathbb{P}_2(F) \to \mathscr{A}_2^0(F)$ given by $(E/F, \xi) \mapsto \pi_\xi$. From this there is an obvious suggestion for what the map $\mathcal{G}_2^0(F) \to \mathscr{A}_2^0(F)$ should be, namely the assignment $\rho_\xi \mapsto \pi_\xi$. We let this be a lesson in the dangers of naivety: The map $\rho_\xi \mapsto \pi_\xi$ does not satisfy the statements of theorem 6.2.1. If it did, then from proposition 6.2.2 the central character of π_ξ should be $\det \rho_\xi$. However, the central character of π_ξ is $\xi|_{F^\times}$ (cf. theorem 3.7.5), whereas one can compute (cf. [BH06, §29.2 Prop.]) that

$$\det \rho_\xi = \varkappa_{E/F} \otimes \xi|_{F^\times},$$

where $\varkappa_{E/F}$ is as in § 4.1.2.

We accordingly make some adjustments to the map $\rho_\xi \mapsto \pi_\xi$ to arrive at the map $\mathbf{LLC}^0$. The plan goes like this: To an admissible pair $(E/F, \xi)$ we will associate a character Δ_ξ of $E^\times$ of level zero. Then, we show that the map $\mathbb{P}_2(F) \to \mathbb{P}_2(F)$ given by $(E/F, \xi) \mapsto (E/F, \Delta_\xi\xi)$ is bijective. The bijection $\mathbf{LLC}^0$ will be given as follows: If $(E/F, \xi)$ is an admissible pair and $\rho_\xi \cong \rho \in \mathcal{G}_2^0(F)$, then $\mathbf{LLC}^0(\rho) = \pi_{\Delta_\xi\xi}$. Thus the first objective ought to be defining Δ_ξ.

6.2.2 The Character Δ_ξ Associated to $(E/F, \xi)$

We begin by defining the *Langlands constant* $\lambda_{E/F}$ for a finite separable extension E of F and a nontrivial additive character ψ of F:

$$\lambda_{E/F}(\psi) = \frac{\varepsilon(R_{E/F}, s, \psi)}{\varepsilon(\mathbb{1}_E, s, \psi_E)},$$

where $R_{E/F}$ is the regular representation.

Proposition 6.2.5. *Let ψ be an additive character of F of level one and let E/F be a tamely ramified extension. If E/F is unramified, then $\varkappa_{E/F}$ is unramified of order 2, and*

$$\lambda_{E/F}(\psi) = -1.$$

If E/F is totally ramified, then $\varkappa_{E/F}$ is the nontrivial character of $F^\times/N_{E/F}(E^\times)$, and

$$\lambda_{E/F}(\psi) = \tau(\varkappa_{E/F}, \psi)q^{-1/2},$$

where τ is the Gauss sum:

$$\tau(\varkappa_{E/F}, \psi) = \sum_{U_F/U_F^{n+1}} \check{\varkappa}_{E/F}(cx)\psi(cx)$$

with n the level of $\varkappa_{E/F}$ and $c \in F$ such that $v_F(c) = -n$. In particular,

$$\lambda_{E/F}(\psi)^2 = \varkappa_{E/F}(-1).$$

Proof. See [BH06, §34.3] □

In defining the character Δ_ξ of $E^\times$, we split the matter into two cases. The first of which is for an admissible pair $(E/F, \xi)$ where E/F is unramified.

Definition 6.2.6. Let $(E/F, \xi)$ be an admissible pair in which E/F is an unramified extension. Then we define Δ_ξ to be unramified of order 2.

The ramified case is more involved. Put μ_F for the group of roots of unity in F of order prime to p. If E/F is a totally ramified quadratic extension (automatically tame by our assumption on residual characteristics) then we have $U_E = \mu_E U_E^1 = \mu_F U_E^1$. It follows then that for any $\beta \in E^\times$ and uniformizer ϖ of E, there exists a unique root of unity $\zeta(\beta, \varpi)$ such that

$$\beta \varpi^{-v_E(\beta)} \equiv \zeta(\beta, \varpi) \pmod{U_E^1}.$$

With this, we have the following:

Proposition 6.2.7 (Proposition-Definition)**.** *Let ψ be an additive character of F of level one and let $(E/F, \xi)$ be a minimal admissible pair with E/F totally ramified. Put n for the level of ξ and let $\alpha \in \mathfrak{p}_E^{-n}$ be such that $\xi(1+x) = \psi_E(\alpha x)$ for all $x \in \mathfrak{p}_E^n$. Then, there is a unique character Δ_ξ such that*

$$\begin{aligned} \Delta_\xi|_{U_E^1} &= 1, \\ \Delta_\xi|_{F^\times} &= \varkappa_{E/F}, \\ \Delta_\xi(\varpi) &= \varkappa_{E/F}(\zeta(\alpha, \varpi))\lambda_{E/F}(\psi)^n \end{aligned}$$

for any uniformizer ϖ of E. The definition of Δ_ξ is independent of choices of α and ψ.

If $(E/F, \xi)$ is any admissible pair with E/F totally ramified, then we write $\xi = \xi' \otimes (\chi \circ N_{E/F})$ for a minimal admissible pair $(E/F, \xi')$ and a character χ of $F^\times$. We then define

$$\Delta_\xi = \Delta_{\xi'}.$$

This definition of Δ_ξ is independent of choice of minimal admissible pair for the decomposition $\xi = \xi' \otimes (\chi \circ N_{E/F})$.

Proof. The properties demanded of the character Δ_ξ ensure its unicity. Moreover, for an additive character ψ of level one the definition of Δ_ξ depends only on the coset αU_E^1. Replacing ψ with some ψ' of level one, we get $\psi' = u\psi$ for $u \in U_F$. Then α is replaced by $u^{-1}\alpha$ and we get

$$\varkappa_{E/F}(\zeta(u^{-1}\alpha, \varpi))\lambda_{E/F}(u\psi)^n = \varkappa_{E/F}(\zeta(\alpha, \varpi))\lambda_{E/F}(\psi)^n \varkappa_{E/F}(u)^{n-1},$$

following from proposition 6.2.5. But the fact that $(E/F, \xi)$ is a minimal admissible pair (in particular that $\xi|_{U_E^n}$ does not factor through $N_{E/F}$) forces that the level of ξ is odd. Thus since $\varkappa_{E/F}$ is a character of order 2, we have $\varkappa_{E/F}(u)^{n-1} = 1$. Therefore we see that the definition of Δ_ξ is independent of our choice of ψ and α.

Thus the proof of the proposition is a matter of showing existence, which boils down to checking that we have

$$\Delta_\xi(u\varpi) = \Delta_\xi(u)\Delta_\xi(\varpi)$$

for $u \in U_E$ and that

$$\Delta_\xi(\varpi^2) = \Delta_\xi(\varpi)^2.$$

For full details of this, see [BH06, §34.4]. □

Let note some consequences of this definition. Firstly, it follows readily that if $(E/F, \xi)$ is an admissible pair, then so too is the pair $(E/F, \Delta_\xi \xi)$. Thus we define a map

$$\mathbb{P}_2(F) \to \mathbb{P}_2(F)$$

via $(E/F, \xi) \mapsto (E/F, \Delta_\xi \xi)$. This map is a bijection, following from the fact that Δ_ξ depends only on the tame quadratic extension E/F and $\xi|_{U_E^1}$.

We form the map $\mathbf{LLC}^0$ as was described earlier: For $\rho \in \mathcal{G}_2^0(F)$, by theorem 6.2.3, there is an admissible pair $(E/F, \xi)$ such that $\rho \cong \rho_\xi = \mathrm{Ind}_{E/F}\,\xi$. Then using the machinery just developed, we associate to ρ the admissible pair $(E/F, \Delta_\xi \xi)$. The supercuspidal representation $\mathbf{LLC}^0(\rho)$ is finally defined to be $\pi_{\Delta_\xi \xi}$, as was constructed in section 3.7.

Theorem 6.2.8. *When the residual characteristic of F is $p \neq 2$, then the map*

$$\mathbf{LLC}^0 \colon \mathcal{G}_2^0(F) \to \mathscr{A}_2^0(F)$$

as described above is a bijection satisfying

(i) $\varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\chi \mathbf{LLC}^0(\rho), s, \psi)$

(ii) $\mathbf{LLC}^0(\chi \otimes \rho) = \chi \mathbf{LLC}^0(\rho)$,

(iii) $\mathbf{LLC}^0(\check{\rho}) = \mathbf{LLC}^0(\rho)^\vee$.

for any character χ of $F^\times$ and any nontrivial additive character ψ of F.

Before proving the theorem, we must state two computational propositions which will be used *ad nauseam*.

Proposition 6.2.9. *Let ξ be a ramified character of $F^\times$ of level $n \geq 0$, and let ψ be an additive character of F of level 1. Then,*

$$\varepsilon(\chi, s, \psi) = q^{n(\frac{1}{2}-s)} \tau(\chi, \psi)/q^{(n+1)/2},$$

where $\tau(\chi, \psi)$ is the Gauss sum as defined in proposition 5.1.4.

And, for computing Gauss sums:

Proposition 6.2.10. *Suppose that ξ is a character of $F^\times$ of level $n \geq 1$ and ψ an additive character of F of level 1. Let $c \in F$ satisfy*

$$\xi(1+x) = \psi(cx)$$

for $x \in \mathfrak{p}^{[n/2]+1}$. Then

$$\tau(\chi, \psi) = q^{[(n+1)/2]} \sum_y \check{\chi}(cy)\psi(cy)$$

with y ranging over $U_F^{[(n+1)/2]}/U_F^{[n/2]+1}$.

Proofs for both of the two propositions may be found in [BH06, §23]. With this, we may proceed.

Proof of theorem 6.2.8. The fact that $\mathbf{LLC}^0$ is a bijection follows from the maps $\rho \mapsto \rho_\xi$, $(E/F, \xi) \mapsto (E/F, \Delta_\xi \xi)$, and $(E/F, \Delta_\xi) \mapsto \pi_{\Delta_\xi \xi}$ all being bijective, as we have observed in our work up to this point. From proposition 6.2.2, it suffices to check condition (i) in the theorem for a single fixed choice of nontrivial additive character ψ. Henceforth we work relative to an additive character ψ of level 1. Moreover, (ii) is true by construction and (iii) is a direct consequence of theorem 3.7.5(iii). Therefore we just need to show the relationship (i) between local epsilon factors.

It follows from proposition 4.1.5(iii), proposition 6.2.2, and theorem 3.7.5(ii) that the integers $n(\rho, \psi)$ and $n(\mathbf{LLC}(\rho), \psi)$ (as appearing in proposition 4.1.5 and § 5.1, respectively) are equal. Thus we need only to show (i) at $s = 1/2$. We first consider the totally ramified case in which there is no twisting to be concerned with:

Case 1: Let $\rho \cong \rho_\xi = \mathrm{Ind}_{E/F}\, \xi$ for $(E/F, \xi)$ a minimal admissible pair and E/F totally ramified. In this case ξ has odd level $n = 2m+1$. We choose α such that $\xi(1+x) = \psi_E(\alpha x)$ for all $x \in \mathfrak{p}_E^{m+1}$. From proposition 4.1.5(iii), we have

$$\varepsilon(\rho, s, \psi) = \varepsilon(\xi, s, \psi_E)\lambda_{E/F}(\psi),$$

where we recall the notation $\psi_E = \psi \circ \mathrm{tr}_{E/F}$. A computation reveals that

$$\varepsilon(\xi, 1/2, \psi_E) = \check{\xi}(\alpha)\psi_E(\alpha),$$

and so

$$\varepsilon(\rho, 1/2, \psi) = \check{\xi}(\alpha)\psi_E(\alpha)\lambda_{E/F}.$$

On the other hand, setting $\pi = \mathbf{LLC}^0(\rho)$, we may use proposition 5.1.4 to calcuate

$$\varepsilon(\pi, 1/2, \psi) = \check{\Delta}_\xi(\alpha)\check{\xi}(\alpha)\psi_A(\alpha).$$

Since E embeds into $A = \mathrm{Mat}_2(F)$, we see that $\psi_E(\alpha) = \psi_A(\alpha)$, so we just need to show that $\check{\Delta}_\xi(\alpha) = \lambda_{E/F}(\psi)$. Let ϖ be a uniformizer of E and $\zeta \in \mu_F$ such that $\alpha\varpi^n \cong \zeta \pmod{U_E^1}$. Then, by how Δ_ξ is defined, we have

$$\Delta_\xi(\alpha) = \Delta_\xi(\zeta)\Delta_\xi(\varpi^{-n}).$$

But $\zeta \in F^\times$, so this is

$$\varkappa_{E/F}(\zeta)\Delta_\xi(\varpi^{-n}) = \varkappa_{E/F}(\zeta)^{1-n}\lambda_{E/F}(\psi)^{-n^2}.$$

Thus we have

$$\Delta_\xi(\alpha)\lambda_{E/F}(\psi) = \varkappa_{E/F}(\zeta)^{1-n}\lambda_{E/F}(\psi)^{1-n^2} = 1,$$

with the last equality holding from the proposition 6.2.5, the fact that $\varkappa_{E/F}$ is of order 2, and that n is odd. Hence we see that $\lambda_{E/F}(\psi) = \check{\Delta}_\xi(\alpha)$. With this settled, we want to understand how twisting by a multiplicative character affects the situation.

Case 2: We let $\rho \cong \rho_\xi$ be as in case 1, and let χ be a character of $F^\times$ of level $l \geq 1$. Now consider the representation $\chi \otimes \rho$, which is obtained by from induction from the admissible pair $(E/F, \xi\chi_E)$. Again, the level of χ is $n = 2m+1$. If $l \leq m$, then the pair $(E/F, \xi\chi_E)$ is minimal and we are in the situation of case 1. Thus we assume that $l > m$, implying that χ_E has level $2l > n$. There then exists $c \in F$ such that $\chi(1+x) = \psi(cx)$ for $x \in \mathfrak{p}_F^{[l/2]+1}$, and so $\chi_E(1+x) = \psi_E(cx)$ for $x \in \mathfrak{p}_E^{l+1}$. As was in case 1, we also have $\xi(1+x) = \psi_E(\alpha x)$ for $x \in \mathfrak{p}_E^l$. With these observations, after proceeding with standard epsilon factor computations using Gauss sums (cf. [BH06, §23.6]), we are able to arrive at

$$\varepsilon(\xi\chi_E, 1/2, \psi_E) = \check{\xi}(c+\alpha)\check{\chi}_E(1+c^{-1}\alpha)\psi_E(\alpha)\varepsilon(\chi_E, 1/2, \psi_E).$$

Employing once more proposition 4.1.5, we have

$$\begin{aligned}\varepsilon(\chi\otimes\rho, 1/2, \psi) &= \check{\xi}(c+\alpha)\check{\chi}_E(1+c^{-1}\alpha)\psi_E(\alpha)\varepsilon(\chi_E, 1/2, \psi_E)\lambda_{E/F}(\psi)\\ &= \check{\xi}(c+\alpha)\check{\chi}_E(1+c^{-1}\alpha)\psi_E(\alpha)\varepsilon(\chi, 1/2, \psi)\varepsilon(\chi\varkappa_{E/F}, 1/2, \psi)\\ &= \check{\xi}(c+\alpha)\check{\chi}_E(1+c^{-1}\alpha)\psi_E(\alpha)\varepsilon(\chi, 1/2, \psi)^2\varkappa_{E/F}(c).\end{aligned}$$

We once again write $\pi = \mathbf{LLC}^0(\rho)$. A standard computation on this side finds

$$\varepsilon(\chi\pi, 1/2, \psi) = \check{\Delta}_\xi\check{\xi}(c+\alpha)\check{\chi}(\det(1+c^{-1}\alpha))\psi_A(\alpha)\varepsilon(\chi, 1/2, \psi)^2.$$

We again observe that $\psi_A(\alpha) = \psi_E(\alpha)$, and that by construction $\Delta_\xi(c+\alpha) = \Delta_\xi(c) = \varkappa_{E/F}(c)$. Therefore in this case we have

$$\varepsilon(\chi\otimes\rho, 1/2, \psi) = \varepsilon(\chi\pi, 1/2, \psi),$$

as desired.

Case 3: The final case which we consider is when $\rho \cong \rho_\xi = \mathrm{Ind}_{E/F}\,\xi$ for $(E/F, \xi)$ a minimal admissible pair with E/F unramified. Let n be the level of ξ. If n is odd, then the computations follow closely to case 1. The case $n = 0$ is handled in [BH06, §25, §35.4]. We accordingly assume $n = 2m > 0$. In this setting,

$$\varepsilon(\xi, 1/2, \psi_E) = q^{-1}\sum_{x\in\mathfrak{p}_E^m/\mathfrak{p}_E^{m+1}}\check{\xi}(\alpha(1+x))\psi_E(\alpha(1+x)).$$

If we let $\pi = \mathbf{LLC}^0(\rho)$, from proposition 5.1.4,

$$\varepsilon(\pi, 1/2, \psi) = q^{-3}\sum_{y\in\mathfrak{p}_F^m\mathfrak{M}/\mathfrak{p}_F^{m+1}\mathfrak{M}}\operatorname{tr}\Lambda^\vee(\alpha(1+y))\psi_A(\alpha(1+y)),$$

where we recall $\mathfrak{M} = \mathrm{Mat}_2(\mathcal{O}_F)$ and Λ is as appears in the construction of π_ξ in section 3.7. The following lemma is a direct consequence of the lemma in [BH06, §22.5].

Lemma 6.2.11. *Let $x \in U_{\mathfrak{M}}^m$. Then $\operatorname{tr}\Lambda(\alpha x) = 0$ unless αx is $U_{\mathfrak{M}}^m$-conjugate to an element of $\alpha U_E^m U_{\mathfrak{M}}^{m+1}$.*

Applying this to our expression for $\varepsilon(\pi, 1/2, \psi)$ and counting the number of elements in conjugacy classes, we get

$$\varepsilon(\pi, 1/2, \psi) = -q^{-1}\sum_{y\in\mathfrak{p}_E^m/\mathfrak{p}_E^{m+1}}\check{\xi}(\alpha(1+x))\psi_A(\alpha(1+x)).$$

Now $\varepsilon(\rho, 1/2, \psi) = \varepsilon(\xi, 1/2, \psi_E)\lambda_E/F(\psi)$. But E/F is unramified, thus by proposition 6.2.5, $\lambda_E/F(\psi) = -1$. Therefore we have

$$\varepsilon(\pi, 1/2, \psi) = \varepsilon(\rho, 1/2, \psi),$$

as desired.

Finally, for representations of the form $\chi\otimes\rho$ for ρ determined by an unramified minimal admissible pair and χ a character of $F^\times$, the computations are essentially the same as case 2. □

We now simply define

$$\mathbf{LLC}\colon \mathcal{G}_2(F) \to \mathscr{A}_2(F)$$

by

$$\mathbf{LLC}(\rho) = \begin{cases} \mathbf{LLC}^0(\rho) & \text{if } \rho \in \mathcal{G}_2^0(F), \\ \mathbf{LLC}^1(\rho) & \text{if } \rho \in \mathcal{G}_2^1(F). \end{cases}$$

From theorems 6.1.1 and 6.2.8, it follows that **LLC** is the unique bijective map of theorem 6.0.1. Therefore the local Langlands correspondence for $\mathrm{GL}_2(F)$ (when $p \neq 2$) is proved.

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