An Introduction to Equivariant Cohomology

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1 Introduction

Cohomology theories are powerful tools in algebraic topology (as well as nearly all facets of math) to transform problems of geometry into algebraic problems. More precisely, cohomology is a functor from topological spaces to rings. Often our topological spaces have natural symmetries captured by a group acting on said space. Equivariant cohomology is a cohomology theory which takes this into account which can simplify various problems.

Suppose a topological group G has a continuous left (respectively right) action on a topological space X. We then say X is a *left (respectively right) G-space*. A continuous map $f\colon X \to Y$ of G-spaces X and Y is *G-equivariant* if

$$f(g \cdot x) = g \cdot f(x)$$

for all $x \in X$, $g \in G$. We also refer to G-equivariant maps as G-maps. The collection of G-spaces and G-maps form a category for a fixed group G. Note that we drop the left/right distinction since any left G-action can be turned into a right action via

$$g \cdot x = x \cdot g^{-1}.$$

If X is a C^∞ manifold with a smooth action of a Lie group G then we say X is a *G-manifold*. Similar to before, smooth G-manifolds and smooth G-maps form a category for a fixed Lie group G. That said, for the scope of this paper we will mostly work in the category of G-spaces.

We will take cohomology to mean singular cohomology with integer coefficients. This is a contravariant functor

$$H^*(-)\colon \mathrm{Top} \to \mathrm{Ring},$$

where we have

$$H^*(M;\mathbb{Z}) = \bigoplus_{n\geq 0} H^n(M;\mathbb{Z}).$$

In this way $H^*(M)$ obtains a graded-commutative ring structure with the cup product as multiplication. Then for a topological group G we will define equivariant cohomology to be a contravariant functor

$$H^*_G(-)\colon \{G\text{-spaces}\} \to \mathrm{Ring}.$$

If we want to define a cohomology theory accounting for the symmetries of the action of G on X, then one may initially think to just take the singular cohomology of X/G. We will see that X/G is somehow "poorly-behaved" if the action of G on X is not free, and hence this construction proves to be not particularly useful. This will lead us to the *Borel Construction*, which will be how we define equivariant cohomology. Our exposition will largely follow that provided in [3], providing more details in certain proofs and explanations where we see appropriate.

2 Recollections on Homotopy and Fiber Bundles

Recall that given two maps between based toplogical spaces $f, g\colon (X, x_0) \to (Y, y_0)$, a *homotopy* between them is a continuous map

$$H\colon X \times I \to Y$$

such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ and $H(x_0, t) = y_0$ for all t. The *fundamental group of X* denoted $\pi_1(X, x_0)$ is the set of homotopy classes of based maps $S^1 \to X$. That is,

$$\pi_1(X, x_0) = [(S^1, s_0), (X, x_0)].$$

As the name suggests, this set has a group structure. This notion generalizes, and we can define the *q-th homotopy group of X* to be

$$\pi_q(X, x_0) = [(S^q, s_0), (X, x_0)].$$

Given a map $f\colon (X, x_0) \to (Y, y_0)$ we get an induced map $f_*\colon \pi_q(X, x_0) \to \pi_q(Y, y_0)$ where we define

$$f_*[\alpha] = [f \circ \alpha]$$

for $[\alpha] \in \pi_q(X, x_0)$. One can readily verify that for $q \geq 1$ this is a well defined group homomorphism (of course, for $q = 0$, $\pi_0(X, x_0)$ is not in general a group). In this way we get a covariant functor

$$\pi_q(-)\colon \mathrm{Top}_* \to \mathrm{Grp},$$

where by Top_* we mean the category of based topological spaces with basepoint-preserving maps as morphisms. A *homotopy equivalence* between spaces (X, x_0) and (Y, y_0) is a map $f\colon (X, x_0) \to (Y, y_0)$ with a homotopy inverse. That is, a map $g\colon (Y, y_0) \to (X, x_0)$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$, where $\sim$ is denoting the relation of homotopy. In the case that there exists a homotopy equivalence between X and Y we say they are of the same *homotopy type.* If two spaces X and Y are of the same homotopy type then we have

$$\pi_q(X, x_0) = \pi_q(Y, y_0)$$

for all q.

Recall a *fiber bundle with fiber F* is a continuous surjection $\pi\colon E \to B$ that is locally trivial. That is, for every $b \in B$, there exists a neighborhood of b, U, such that $\pi^{-1}(U) \cong U \times F$. Our main concern with such gadgets will be the related exact sequence of homotopy groups.

Proposition 1. *Let $p\colon (E, x_0) \to (B, b_0)$ be a fiber bundle of based spaces (i.e. $p(x_0) = b_0$) with fiber $F = p^{-1}(b_0)$ and path connected base space B. Since F includes into E we can take F as based space with basepoint x_0 as well. Let $\iota\colon (F, x_0) \to (E, x_0)$ be the inclusion map. Then, there is a long exact sequence of homotopy groups:*

$$\cdots \to \pi_q(F, x_0) \xrightarrow{\iota_*} \pi_q(E, x_0) \xrightarrow{p_*} \pi_q(B, b_0) \to \pi_{q-1}(F, x_0) \to \cdots$$

Proving the existence of such a sequence and its exactness requires developing some homotopy theory that is not particularly pertinent to the content we wish to develop in this paper. For that reason we will omit the proof of proposition 1, however we remark that this is a standard result of algebraic topology and proofs can be found in nearly any text on the subject, e.g. [1].

In order to define equivariant cohomology we will need to consider a special kind of fiber bundle: a *principal G-bundle.*

Definition 1 (Principal G-Bundle)**.** Let G be a topological group. A *principal G-bundle* is a fiber bundle $p\colon E \to B$ with fiber G such that

1. G has a free right action on P.
2. The local triviality homeomorphism $\varphi_U : p^{-1}(U) \xrightarrow{\sim} U \times G$ is G-equivariant, with G acting on $U \times G$ on the right via $(u, x) \cdot g = (u, xg)$.

The following proposition yields us a bounty of examples:

Proposition 2. *If a compact Lie group G acts smoothly and freely on a manifold M, then the projection map $\pi\colon M \to M/G$ is a smooth principal G-bundle.*

Example 1. The action of S^1 on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by rotation is smooth and free. Then, S^{2n+1}/S^1 is a definition of $\mathbb{CP}^n$. Thus we see the projection $S^{2n+1} \to \mathbb{CP}^n$ is a principal S^1-bundle.

3 Defining Equivariant Cohomology

Recall in the introduction we discussed a naive approach to defining equivariant cohomology. Indeed if one wishes to "take into account" a group G's action on one's space X, then a natural thought is to just take singular cohomology of the orbit space X/G. However we noted that, unless G acts freely, this orbit space is not generally well behaved. In this section we will first demonstrate by example why that might be the case, then explain how we get around that obstruction.

Take for example the action of S^1 on S^2 by rotation. This is not a free action, as the poles of S^2 are both fixed points of any $x \in S^1$ acting on S^2. The quotient space S^2/S^1 is homeomorphic to the interval I. Then, since I is homotopy equivalent to a point, $H^*(S^2/S^1) = H^*(\mathrm{pt})$. However, there is interesting information in the S^1 action on S^2 that is lost if we take this as our definition of equivariant cohomology. Our goal then is, given an arbitrary space M and a group G acting on M, to obtain a free action of G while not modifying the homotopy type of M. We accomplish this by constructing the *homotopy quotient* of M by G. First consider the following lemma.

Lemma 1. *Let E be a space on which G acts freely. Then no matter how G acts on M, the diagonal action of G on $E \times M$ via $g \cdot (e, m) = (g \cdot e, g \cdot m)$ is free.*

Proof. We have $g \cdot (e, m) = (g \cdot e, g \cdot m) = (e, m)$ if and only if $g \cdot e = e$ and $g \cdot m = m$. Since G acts freely on E, this holds only if $g = \mathbb{1}_G$. Thus G acts freely on $E \times M$. □

Definition 2 (Homotopy Quotient)**.** Let EG be a contractible space on which G acts freely, and M any space with a G action. Then the *homotopy quotient of M by G*, denoted by M_G, is the orbit space of $EG \times M$ by the diagonal action of G.

Observe that, by lemma 1, $EG \times M$ has a free action of G. Since EG is contractible, we then have

$$\pi_q(EG \times M) = \pi_q(EG) \times \pi_q(M) = 0 \times \pi_q(M) \cong \pi_q(M).$$

Thus, $EG \times M$ is of the same homotopy type as M. We are now able to define equivariant cohomology.

Definition 3 (Equivariant Cohomology)**.** Let M be a space with a G action. The *equivariant cohomology of M by G* is the singular cohomology of the homotopy quotient M_G:

$$H^*_G(M) = H^*(M_G).$$

There is a pressing issue of well-definedness here. Indeed we ostensibly had a choice of the contractible space with a free G action EG. The remainder of this section will be devoted to showing this equivariant cohomology is in fact a well defined notion. We first introduce a generalization of the homotopy quotient defined earlier.

Definition 4 (Borel Construction)**.** Let P be a right G-space and M a left G-space. We then define the *mixing space* of P and M to be the quotient of $P \times M$ by the relation $(p, m) \sim (p \cdot g, g^{-1} \cdot m)$ for some $g \in G$. We write

$$P \times_G M = P \times G/\sim .$$

The equivalence class of (p, m) in $P \times_G M$ is denoted by $[p, m]$, as usual.

Let $f\colon P \to P'$ be a right G-map. Define $\mathcal{F}_M(f)\colon P \times_G M \to P' \times_G M$ by $f[p, m] = [f(p), m]$. This is well defined by f being a G-map. Then, letting $\mathcal{F}_M(P) = P \times_G M$ we get a covariant functor

$$\mathcal{F}_M(-)\colon \{\text{Right } G\text{-spaces}\} \to \text{Top}.$$

This Borel construction functor is key to showing equivariant cohomology is well defined. In particular, it fits into a commutative diagram called *Cartan's mixing diagram.*

Proposition 3. *Let $\alpha\colon P \to B$ be a principal G-bundle and define $\tau_1\colon P \times_G M \to B$ via $\tau_1[p, m] = \alpha(p)$. Then τ_1 is a fiber bundle with fiber M.*

Before proving the proposition, we will show the mixing diagram.

$$\begin{array}{ccccc} P & \xleftarrow{\pi_1} & P \times M & \xrightarrow{\pi_2} & M \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ B & \xleftarrow{\tau_1} & P \times_G M & \xrightarrow[\tau_2]{} & M/G \end{array}$$

Above π_1, π_2, β, and γ are all the natural projections. The map τ_2 is given by

$$[p, m] \mapsto Gm.$$

We will return to this diagram often in proving things about equivariant cohomology.

Proof of proposition 3. Suppose U is a neighborhood in B such that $\alpha^{-1}(U) \cong U \times G$. Then we wish to show that ${\tau_1}^{-1}(U) \cong U \times M$. Then every point in B will have a locally trivial neighborhood. We have

$${\tau_1}^{-1}(U) = \{[p, m] \in P \times_G M \mid \tau_1[p, m] = \alpha(p) \in U\} = \{[p, m] \in P \times_G M \mid p \in \alpha^{-1}(U)\}.$$

But this is just $\alpha^{-1}(U) \times_G M$. We assume that $\alpha^{-1}(U) \cong U \times G$ and so by the functoriality of $\mathcal{F}_M(-)$,

$$\alpha^{-1}(U) \times_G M \cong (U \times G) \times_G M.$$

For any topological space U we have a homeomorphism

$$(U \times G) \times_G M \xrightarrow{\sim} U \times M$$

given by

$$[(u, g), m] \mapsto (u, gm)$$

with inverse

$$(u, m) \mapsto [(u, 1), m].$$

Thus we have ${\tau_1}^{-1}(U) \cong U \times M$. □

Returning the the mixing diagram, we remark that by a symmetrical argument if $\gamma\colon M \to M/G$ is a principal G-bundle then the map $\tau_2\colon P \times_G M \to M/G$ is a fiber bundle with fiber P. With propositon 3 we can regard the functor $\mathcal{F}_M(-)$ with a tad more specificity. We now have a covariant functor

$$\mathcal{F}_M(-)\colon \{\text{Principal } G\text{-bundles over } B\} \to \{\text{Fiber bundles over } B \text{ with fiber } M\}.$$

Lemma 2. *If E is a weakly contractible space with a topological group G action and $P \to P/G$ is a principal G bundle, then $(E \times P)/G$ and P/G are weakly homotopy equivalent, in the sense that $\pi_q((E \times P)/G) \cong \pi_q(P/G)$ for all q.*

Proof. Consider the mixing diagram.

$$\begin{array}{ccccc} E & \xleftarrow{\pi_1} & E \times P & \xrightarrow{\pi_2} & P \\ \alpha\big\downarrow & & \beta\big\downarrow & & \big\downarrow\gamma \\ E/G & \xleftarrow{\tau_1} & E \times_G P \cong (E \times P)/G & \xrightarrow[\tau_2]{} & P/G \end{array}$$

Then by proposition 3, τ_2 is a fiber bundle with fiber E. By proposition 1, we have a long exact sequence

$$\cdots \to \pi_q(E) \to \pi_q((E \times P)/G) \to \pi_q(P/G) \to \pi_{q-1}(E) \to \cdots$$

But we assumed that E is weakly contractible, so $\pi_q(E) = 0$ for all q. Thus the sequence being exact requires that

$$\pi_q((E \times P)/G) \xrightarrow{\sim} \pi_q(P/G)$$

be an isomorphism. □

The following theorem will allow us to prove well-definedness:

Theorem 1. *Let G be a topological group acting on the left on a space M. If $E \to B$ and $E' \to B$ are two principal G-bundles with weakly contractible total spaces, then $E \times_G M$ and $E' \times_G M$ are weakly homotopy equivalent.*

Proof. We first claim that the projection $E \times M \to (E \times M)/G$ is a principal G-bundle. The principal G-bundle $E \to B$ locally looks like $U \times G \to U$ and so $E \times M \to (E \times M)/G$ locally is $U \times G \times M \to (U \times G) \times_G M$. In the proof of proposition 3, we saw that $(U \times G) \times_G M \cong U \times M$. Then, $U \times G \times M \to (U \times G) \times_G M \cong U \times M$ given via $(u, g, m) \mapsto (u, gm)$ is a trivial bundle with fiber G. Thus $E \times M \to (E \times M)/G$ is a principal G-bundle.

Then, since E' is weakly contractible with a G action, by lemma 2 and the claim we just proved, we have that $(E' \times E \times M)/G$ is weakly homotopy equivalent to $(E \times M)/G$. Analogously, we see that $(E \times E' \times M)/G$ is weakly homotopy equivalent to $(E' \times M)/G$. Clearly the spaces $(E' \times E \times M)/G$ and $(E \times E' \times M)/G$ are homeomorphic, and so we get that $(E \times M)/G$ and $(E' \times M)/G$ are weakly homotopy equivalent, as desired. □

A standard result of algebraic topology is that weak homotopy equivalences between spaces yield isomorphisms between their cohomology groups. Thus we get an isomorphism

$$H^*(E \times_G M) \xrightarrow{\sim} H^*(E' \times_G M).$$

Therefore $H^*_G(M)$ is well-defined.

One upshot of equivariant cohomology is that we can obtain some extra algebraic structure on the ring $H^*_G(M)$ as opposed to $H^*(M)$. Suppose G acts on M and N on the left. Then a G-map $f\colon M \to N$ induces a map $f_G\colon M_G \to N_G$ of homotopy quotients. This map is defined by

$$f_G[e, m] = [e, f(m)].$$

This is well-defined by way of f being a G-map. Take M to be a G-space and consider the constant map

$$\pi\colon M \to \mathrm{pt}.$$

This map is trivially a G-map and so we get an induced map

$$\pi_G\colon M_G \to \mathrm{pt}_G.$$

By the functoriality of $H^*(-)$ we get a homomorphism on singular cohomology graded rings

$$\pi_G^*\colon H^*(\mathrm{pt}_G) \to H^*(M_G) = H_G^*(M).$$

The homotopy quotient of a point by G is what we call the *classifying space* of G, and we write BG. Here we see that $H_G^*(M)$ has the structure of a graded algebra over $H^*(BG)$.

Recall we defined the homotopy quotient of a space M by G in terms of a weakly contractible space EG which had a free G action. A natural question one may ask is how do we know such a space EG exists for a topological group G? It turns out that a weakly contractible space on which G acts freely is precisely the total space of a *universal bundle*.

Definition 5 (Universal Bundle)**.** A principal G-bundle $p\colon EG \to BG$ is a *universal bundle* if

1. for any principal G-bundle $p\colon P \to X$ over a CW-complex X, there exists a continuous map $h\colon X \to BG$ such that $P \cong h^*EG$,
2. if $h_0, h_1\colon X \to BG$ are continuous and $h_0^*EG \cong h_1^*EG$ then h_0 is homotopic to h_1.

As mentioned earlier, the base space BG is called *the classifying space of* G.

Remark 1. Denote by $\mathcal{P}_G(X)$ the set of isomorphism classes of principal G-bundles over X. Then the map

$$\varphi\colon [X, BG] \to \mathcal{P}_G(X)$$

given by

$$h \mapsto h^*EG$$

is a bijection. In this way we can classify principal G-bundles over X by homotopy classes of maps $X \to BG$. While this is not particularly important for our purposes, it provides an explanation for the name "classifying space". Also, the space BG is unique up to homotopy equivalence, and so we can talk about *the* classifying space of G.

Theorem 2 (Steenrod's Criterion)**.** *Let $p\colon E \to B$ be a principal G-bundle with B a CW-complex. Then E is weakly contractible if and only if p is a universal bundle.*

Steenrod's criterion shows us that indeed the universal bundles give us contractible spaces with free G actions (a theorem of Whitehead says that weakly contractible is sufficient for CW-complexes. Universal bundles EG for a topological group G have a CW-complex structure). Here we will detail Milnor's construction of EG for an arbitrary topological group G as he described in his paper [2].

Definition 6 (Join)**.** Given two topological spaces A and B, the *join of A and B* is

$$A * B = (A \times B \times I)/\sim,$$

where $(a, b_1, 0) \sim (a, b_2, 0)$ for all $a \in A$, $b_1, b_2 \in B$ and $(a_1, b, 1) \sim (a_2, b, 1)$ for all $a_1, a_2 \in A$, $b \in B$.

The key observation of the join operation is that if A is n-connected (in the sense that $\pi_q(A) = 0$ for all $q \leq n$) and B is m-connected, then $A * B$ is $(m + n + 2)$-connected. Thus we can obtain higher and higher connected spaces in this way. Furthermore, if both A and B have G actions, say, on the right, then $A * B$ does too:

$$[a, b, t] \cdot g = [a \cdot g, b \cdot g, t].$$

To obtain EG from this, we simply take

$$EG = \varinjlim_{n \in \mathbb{N}} (\underbrace{G * G \cdots * G}_{n \text{ times}}).$$

Since each iteration is increasingly connected, in the limit we have a weakly contractible space. Furthermore, since G acts on itself freely, the resulting space EG has a free G action.

4 Spectral Sequences, Computing Equivariant Cohomology

Up to this point we have defined equivariant cohomology and shown our definition is a good one. Of course the next thing one wishes for is to be able to compute these equivariant cohomology rings. In order to compute an example we first develop some of the theory on spectral sequences, which will be our primary tool for such computations.

Definition 7 (Differential Group)**.** A *differential group* is a pair (E, d) where E is an abelian group and $d\colon E \to E$ is a group homomorphism such that $d^2 = 0$. The *cohomology of* (E, d) is denoted by $H^*(E)$ and is defined to be $\ker(d)/\operatorname{im}(d)$.

Note that the requirement that $d^2 = 0$ ensures $\operatorname{im}(d) \subseteq \ker(d)$ always and so the definition of $H^*(E)$ is okay.

Definition 8 (Spectral Sequence)**.** A *spectral sequence* is sequence of differential groups $\{(E_k, d_k)\}_{k=0}^{\infty}$ such that each E_k is the cohomology of E_{k-1}:

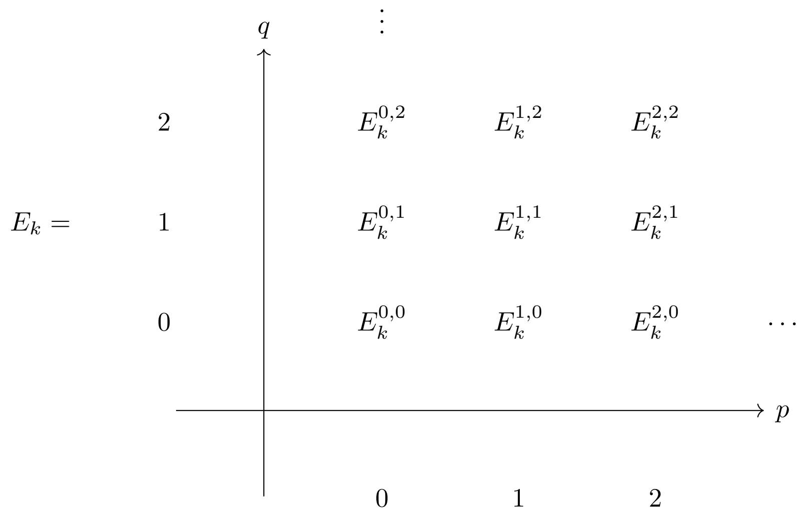
$$E_k = H^*(E_{k-1}).$$

The k-th term of the spectral sequence is referred to as the E_k*-page* of the sequence. We assume that the spectral sequence is *bi-graded*:

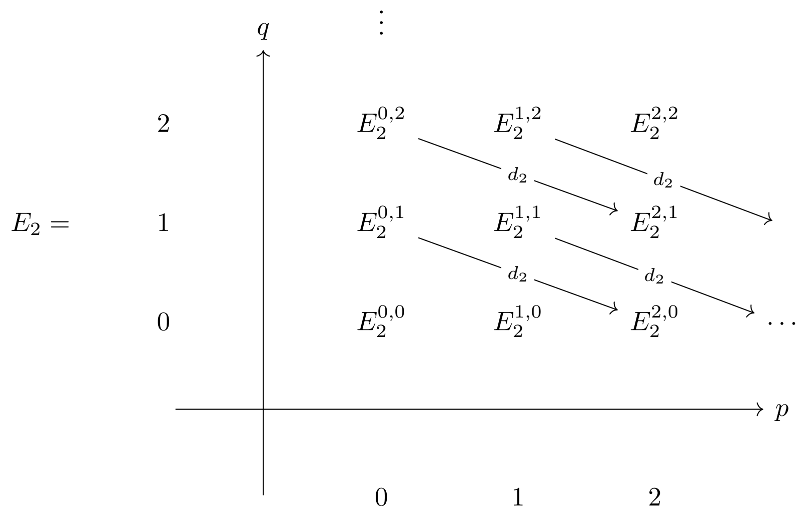
$$E_k = \bigoplus_{p,q \in \mathbb{Z}} E_k^{p,q},$$

and that each differential d_k has degree $(k, -k+1)$ with respect to this bi-grading. We also assume our spectral sequence is a *first-quadrant sequence* in the sense that $E_k^{p,q} = 0$ if $p < 0$ or $q < 0$. With this, the reason for calling each term a "page" becomes apparent: Each term E_k can be visualized as the "page" of a book, where we have boxes for each group in the bi-grading of E_k. Then, as we progress through the spectral sequence, it is as if we are flipping through the pages of the book.

Below we have a picture of what this looks like:



The degree of the differentials can be seen in this way too. On the E_2-page, for example, we have



The above pictorial representation also helps us to see that taking cohomology preserves our bigrading. That is, the (p, q) box of the E_{k+1} page is

$$E_{k+1}^{p,q} = \frac{\ker(d_k \colon E_k^{p,q} \to E_k^{p+k,q-k+1})}{\operatorname{im}(d_k \colon E_k^{p-k,q+k-1} \to E_k^{p,q})}. \tag{1}$$

From this we can also discuss *stationary values.* We assumed that our spectral sequence is a first-quadrant sequence. If we fix some (p, q) with $p, q \geq 0$ we can see for k sufficiently large ($k \geq q + 2$),

$$d_k \colon E_k^{p,q} \to E_k^{p+k,q-k+1}$$

will send $E_k^{p,q}$ to outside of the first quadrant. Thus the kernel of d_k is all of $E_k^{p,q}$. Similarly, if $k \geq p + 1$, then the domain of

$$d_k \colon E_k^{p-k,q+k-1} \to E_k^{p,q}$$

lies outside of the first quadrant and so the image of the above is just 0. Thus, if $k \geq \max(p+1, q+2)$, taking cohomology as in (1) yields

$$E_k^{p,q} = E_{k+1}^{p,q} = E_{k+2}^{p,q} = \cdots$$

For a given (p, q) we denote this stationary value as $E_\infty^{p,q}$.

Now let M be an abelian group. A *filtration* on M is a decreasing sequence of subgroups

$$M = D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots$$

Given a filtration on M we can form the *associated graded group* GM:

$$GM = \bigoplus_{i=0}^{\infty} D_i/D_{i+1}.$$

For a bi-graded first-quadrant page E_k we have a natural *filtration by* p:

$$F_p = \bigoplus_{i\geq p} \bigoplus_{q\geq 0} E_k^{p,q}.$$

With this we may state the main theorem which will allow us to compute equivariant cohomology.

Theorem 3 (Leray)**.** *Let $p\colon E \to B$ be a fiber bundle with fiber F and assume that B is simply connected. Further assume that in every dimension $H^n(F)$ is free of finite rank. Then there exists a spectral sequence with*

$$E_2^{p,q} = H^p(B) \otimes H^q(F)$$

and the filtration on E_2 by p induces a filtration $\{D_p\}$ on $H^*(E)$ such that for each n we have

$$H^n(E) = D_0 \cap H^n(E) \supseteq D_1 \cap H^n(E) \supseteq D_2 \cap H^n(E) \supseteq \cdots$$

where

$$\frac{D_i \cap H^n(E)}{D_{i+1} \cap H^n(E)} = E_\infty^{i,n-i}.$$

We now will demonstrate how we can use theorem 3 to compute equivariant cohomology by considering the example (which was mentioned earlier as to why just taking the singular cohomology of the orbit space might not be a good definition) of S^2 with an S^1 action by rotation.

First we observe that we can regard the homotopy quotient M_G as a fiber bundle. Suppose G is a topological group who acts on a space M. Let $EG \to BG$ be a universal bundle. Then, returning to Cartan's mixing diagram, we observe that $\tau_1\colon M_G \to BG$ is a fiber bundle with fiber M by proposition 3.

$$\begin{array}{ccccc} EG & \longleftarrow & EG \times M & \longrightarrow & M \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow & & \big\downarrow \\ BG & \underset{\tau_1}{\longleftarrow} & M_G & \longrightarrow & M/G \end{array}$$

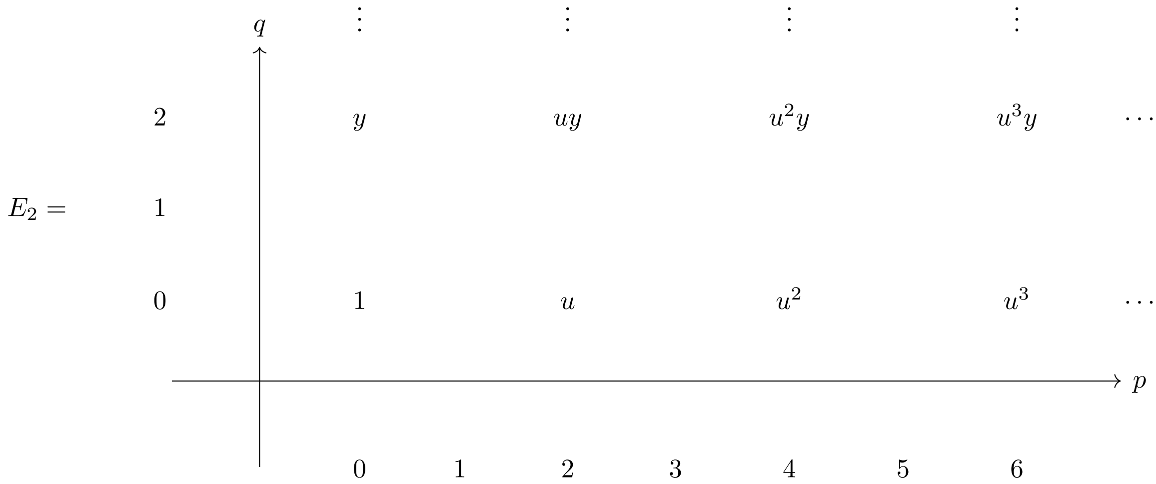
Therefore $(S^2)_{S^1}$ is a fiber bundle over $BS^1 = \mathbb{CP}^\infty$ with fiber S^2.

Remark 2. To see that $BS^1 = \mathbb{CP}^\infty$, recall a definition we took for BG is pt_G. Thus $BG = (EG \times \mathrm{pt})/G \cong EG/G$. A weakly contractible space with a free S^1 action is S^∞. Thus we have $BS^1 \cong S^\infty/S^1 = \mathbb{CP}^\infty$.

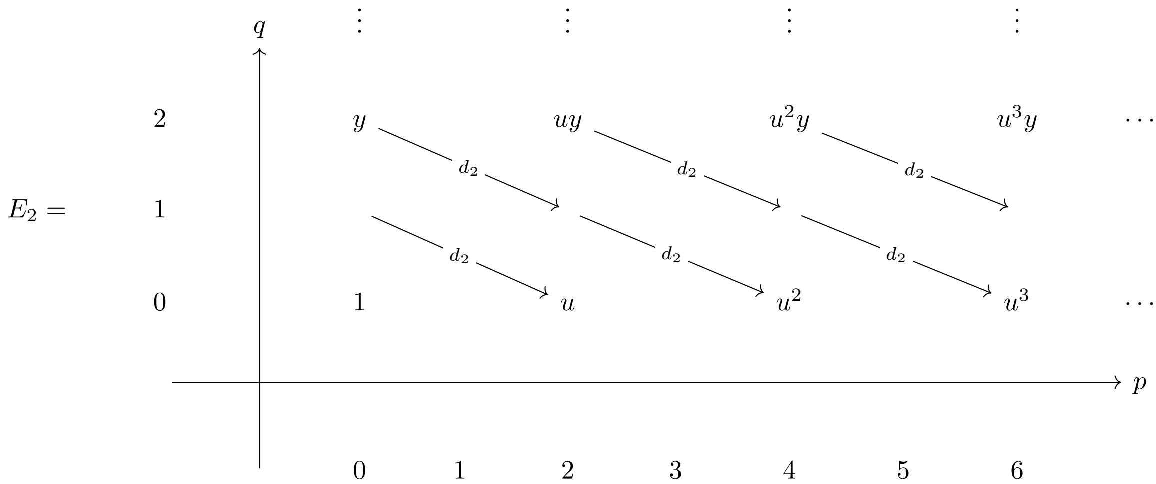
Thus by theorem 3 there exists a spectral sequence with its E_2-page given by

$$E_2 = H^*(\mathbb{CP}^\infty) \otimes_{\mathbb{Z}} H^*(S^2).$$

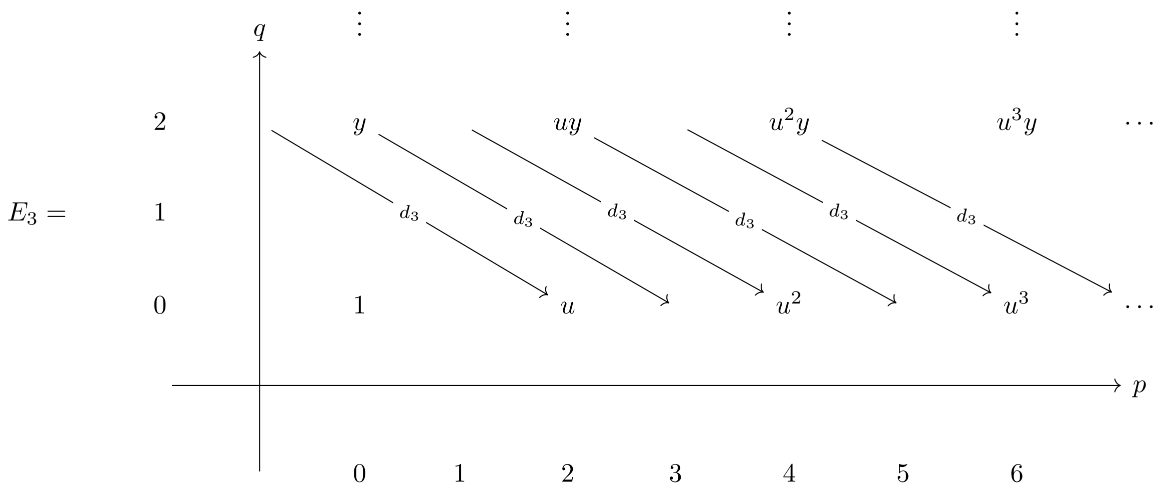
It is a standard result that $H^*(S^2) = \mathbb{Z}[y]/(y^2)$ with $\deg(y) = 2$. One can use theorem 3 to compute $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[u]$ with $\deg(u) = 2$, however we will take this as a fact for brevity's sake. With this, we can model the E_2 page as



As above, boxes that are not filled in are trivial groups. Examining the d_2 differential, we see that the degree of the differentials yields d_2 as always being the 0 map:



Thus by our previous discussion on stationary values, we have $E_2 = E_3$. We can similarly analyze the d_3 differentials:



Again, these are all just the 0 map and so $E_3 = E_4$. For $k \geq 4$, the differentials will land outside of the first quadrant, and so they are all 0 trivially. Therefore we in fact have

$$E_2 = E_3 = \cdots = E_\infty.$$

By our theorem 3, $E_\infty = GH^*((S^2)_{S^1})$. On the 2nd degree, we have a filtration

$$H^2((S^2)_{S^1}) = H^2_{S^1}(S^2) = D_0 \cap H^2_{S^1}(S^2) \supseteq D_1 \cap H^2_{S^1}(S^2) \supseteq D_2 \cap H^2_{S^1}(S^2) \supseteq 0$$

such that

$$\frac{D_i \cap H^2_{S^1}(S^2)}{D_{i+1} \cap H^2_{S^1}(S^2)} = E_\infty^{i,n-i} = E_2^{i,n-i}.$$

From the above pages we have $E_2^{0,2} = \mathbb{Z}y$, $E_2^{1,1} = 0$, and $E_2^{2,0} = \mathbb{Z}u$. Thus $D_1 = D_2 = \mathbb{Z}u$. We then have a short exact sequence

$$0 \to D_1 \to D_0 \to D_0/D_1 \to 0.$$

As above, from the theorem, $D_0 = H^2_{S^1}(S^2)$ and $D_0/D_1 = \mathbb{Z}y$. Thus this sequence is

$$0 \to \mathbb{Z}u \to H^2_{S^1}(S^2) \to \mathbb{Z}y \to 0.$$

Since $\mathbb{Z}y$ is free, this is a split short exact sequence and hence $H^2_{S^1}(S^2) = \mathbb{Z}u \oplus \mathbb{Z}y$. We can play this game again on the at the 4th degree, arriving at $H^4_{S^1}(S^2) = \mathbb{Z}u^2 \oplus \mathbb{Z}uy$. In general, for any positive even degree, we have

$$H^{2n}_{S^1}(S^2) = \mathbb{Z}u^n \oplus \mathbb{Z}u^{n-1}y.$$

For $H^0_{S^1}(S^2)$, we see this is just $\mathbb{Z}$ by looking at the $(0,0)$ box in the page above. Observe in our $E_2 = E_\infty$ page that if $p+q$ is odd then $E_\infty^{p,q} = 0$. Thus, by the same analysis we just performed to compute the even degree groups, $H^{2n+1}_{S^1}(S^2) = 0$. Therefore,

$$H^*_{S^1}(S^2) = \mathbb{Z} \oplus \bigoplus_{n \geq 1} H^{2n}_{S^1}(S^2) \cong \mathbb{Z}[u] \oplus \mathbb{Z}[u]y,$$

with $\deg(u) = \deg(y) = 2$. Here we have recovered the structure of $H^*_{S^1}(S^2)$ as a $\mathbb{Z}[u]$-module. To understand the ring structure, then we need to understand how the y multiplies. The symbol y is degree 2 in $H^2(S^2)$ and so there is some corresponding element $x \in H^2_{S^1}(S^2)$. Thus we wish to know how to multiply x with itself.

Since x is homogeneous of degree 2, $x^2 \in H^4_{S^1}(S^2)$. Thus

$$x^2 = au^2 + bux$$

for some $a, b \in \mathbb{Z}$. With this, we can describe $H^*_{S^1}(S^2)$ as a ring as

$$H^*_{S^1}(S^2) = \mathbb{Z}[u,x]/(x^2 - au^2 - bux).$$

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