# Rigid Analytic Geometry

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## 1 Introduction

In complex algebraic geometry, the analytification functor, which associates to a finite type scheme over $\mathbb{C}$ a complex analytic space (and in particular a complex manifold if one's scheme is smooth) is a tool of great importance. This is because the analytification functor preserves many of the geometric properties of schemes over $\mathbb{C}$ that one cares about. If instead one works over a nonarchimedean local field $K$, we ask if there is a similar method of analytification? One option is base changing to an algebraically closed field, then using Lefschetz principles to use complex analytic methods. The issue here is that one loses arithmetic information in this process.

We want to have a similar functor in the nonarchimedean setting. That is, given a variety over a nonarchimedean local field $K$, can we get an associated "nonarchimedean analytic manifold"? And moreover, in, for example the $p$-adic setting, we want our $p$-adic analytic manifolds to preserve the arithmetic information that a variety over $\mathbb{Q}_p$ would hold. The naive approach would be to define a nonarchimedean analytic manifold in an analogous way to real or complex manifolds: A topological space covered with charts such that it is locally isomorphic to $\mathbb{Q}_p^n$. The issue is that we want to be able to do geometry on these spaces. By "do geometry" we mean have a well-behaved sheaf of analytic functions on our space. The totally disconnected topology on nonarchimdean fields sabotages this[1].

To solve this Tate developed his theory of *rigid analytic spaces*. The construction feels very much like how varieties are developed in classical algebraic geometry. Tate was originally motivated when studying elliptic curves over $\mathbb{C}_p$. He wanted to a uniformization theory similar to that of elliptic curves over $\mathbb{C}$, but as we mentioned, did not want to lose the arithmetic content of a variety over $\mathbb{C}_p$. When he began thinking about this, mathematician Alexander Grothendieck was famously skeptical. In a 1959 letter to Serre, Grothendieck wrote, "Tate has written to me about his elliptic curves stuff, and has asked me if I had any ideas for a global definition of analytic varieties over complete valuation fields. I must admit that I have absolutely not understood why his results might suggest the existence of such a definition, and I remain skeptical. Nor do I have the impression of having understood his theorem at all; it does nothing more than exhibit, via brute formulas, a certain isomorphism of analytic groups; one could conceive that other equally explicit formulas might give another one which would be no worse than his (until proof to the contrary!)" ([Con08]).

Nonetheless, Tate was able to define the appropriate Grothendieck topology and sheaf of "analytic functions" on a space with which we have analytification statements analogous to Serre's GAGA and Chow's theorem. Tate began to lecture on his work at Harvard in the early 60's, but did not publish his findings, instead opting to distribute his notes to his contemporaries. It was not until the notes fell into the hands of the editors at *Inventiones Mathematicae* that they would be published in 1971 ([Tat71]). From there, various other theories tackling a similar problem of analytification of schemes over a nonarchimedean field arose. Among them are Huber's adic spaces, the Berkovich spaces of

[1] A theory defined in this way is not moot! $p$-adic manifolds in this sense to have applications (cf. [Sch11])

Vladimir Berkovich, in more recent times Scholze's work on diamonds and perfectoid spaces, as well others not mentioned. Here we will follow Tate's original approach.

In this paper we will assume an understanding of the basic theory of nonarchimedean valued fields at the level of, say, the first two chapters in [SG13]. In addition, a familiarity algebraic geometry, and in particular varieties, is required. To this end, [Har77] is a great reference.

## 2 The Tate Algebra

Fix once and for all $(K, |\cdot|)$ a nonarchimedean local field with absolute value $|\cdot|$. We will denote by $\mathcal{O}_K$ its valuation ring, with maximal ideal $\mathfrak{p}$ generated by a uniformizing element $\pi$. The residue field $\mathcal{O}_K/\mathfrak{p}$ is denoted by $\kappa$. As we work towards building up to rigid analytic spaces, we should keep in mind its similarities in the theory of classical algebraic geometry by way of varieties. The coordinate ring of an affine variety over a field $k$ who is cut out by polynomials $f_1, \ldots, f_m$ in $n$ variables is the $k$-algebra

$$\frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)}.$$

We want an analog of the polynomial ring $k[x_1, \ldots, x_n]$ in our setting. Moreover, much of the theory behind classical algebraic geometry is reliant on the polynomial ring having certain desirable qualities, like being Noetherian and a unique factorization domain. It is reasonable, then, to ask the same of whatever analog we are to build. If we think of $k[x_1, \ldots, x_n]$ as being the functions on $\operatorname{Spec_m} k[x_1, \ldots, x_n] = \mathbb{A}^n_k$, then the algebra we will define should also contain analytic functions on $K$. Denote by $\mathbb{B}_1(0)$ the closed unit disk in $K$.

**Lemma 2.1.** *Let $(c_i)$ be a sequence of elements in $K$. Then*

$$f = \sum_{i=0}^{\infty} c_i x^i \in K[[x]]$$

*converges on $\mathbb{B}_1(0)$ if and only if $(c_n) \to 0$.*

*Proof.* This follows from the standard similar result for series $\sum_{i=0}^{\infty} c_i$ for $c_i \in K$. □

**Definition 2.2.** The *Tate algebra in $n$ variables* is defined as

$$T_{n,K} = K\langle x_1, \ldots, x_n \rangle = \left\{ f = \sum_{i \in \mathbb{Z}^n_{\geq 0}}^{\infty} c_i x^i \mid (c_i) \xrightarrow{|i| \to \infty} 0 \right\}$$

*Note:* Above by $|i|$ for the $n$-tuple $i$ we mean $|(i_1, \ldots, i_n)| = i_1 + \cdots + i_n$. Furthermore $x^i$ is understood as $x^{i_1} x^{i_2} \cdots x^{i_n}$.

Putting $\mathcal{O}_K\langle x_1, \ldots, x_n \rangle = T_{n,K} \cap \mathcal{O}_K[[x_1, \ldots, x_n]]$, we observe there is a natural reduction map

$$\operatorname{red}\colon \mathcal{O}_K\langle x_1, \ldots, x_n \rangle \to \kappa[x_1, \ldots, x_n].$$

Through lemma 2.1, we see that also any element $f \in T_{n,K}$ defines an evaluation map on $\mathbb{B}^n_1(0)$ to $K$, given by

$$\operatorname{ev}_f \colon \alpha \mapsto c_i \alpha^i,$$

if $f = \sum_{i \in \mathbb{Z}^n_{\geq 0}} c_i x^i$.

Let us now motivate this definition some. By $K^a$ we understand an algebraic closure of $K$. Then it is known that the absolute value on $K$ extends in a unique way to $K^a$, so we can talk about the closed unit ball in $K^a$, written $\mathbb{B}^n_1(0, K^a)$. Denote by $G_K$ the absolute Galois group $G(K^a/K)$.

**Proposition 2.3.** *There is a bijection*

$$\operatorname{Spec}_{\mathrm{m}} T_{n,K} \to \mathbb{B}_1^n(0, K^a)/G_K,$$

*defined by sending associating to $\alpha \in \mathbb{B}_1^n(0, K^a)$ the kernel of the map $T_{n,K} \to K^a$ given by $f \mapsto \mathrm{ev}_f(\alpha)$.*

*Proof.* See ([BGR12], §7.1.1). □

In particular, if $K = K^a$ then $\operatorname{Spec}_{\mathrm{m}} T_{n,K}$ is in bijection with the points of $\mathbb{B}_1^n(0)$. Again relating to the varietal milieu, if $k$ is algebraically closed then $\operatorname{Spec}_{\mathrm{m}} k[x_1, \ldots, x_n] = \mathbb{A}_k^n$ has points in bijection with $k^n$. Thus viewing elements $f \in T_{n,K}$ as functions on $\mathbb{B}_1^n(0)$ is essentially the same as considering them as functions on $\operatorname{Spec}_{\mathrm{m}} T_{n,K}$, as one views the elements of the coordinate ring of an affine variety.

Now that we are convinced that the Tate algebra is a decent analog to the polynomial ring over a field, at least in the sense of its maximal spectrum corresponding affine space, we work to show that $T_{n,K}$ has those nice algebraic qualities of which we spoke earlier. More precisely, we show that $T_{n,K}$ is Noetherian, a unique factorization domain, and Jacobson. We will do this by obtaining a version of the Weierstrass preparation and division theorems from complex analysis. To this end we develop some tools from functional analysis to use on $T_{n,K}$.

Observe that if one completes $\mathcal{O}_K\langle x_1, \ldots, x_n\rangle$ with respect to the $\pi$-adic topology and inverts $\pi$, then one obtains something isomorphic to $T_{n,K}$. This suggests that $T_{n,K}$ is in some sense complete. Let us make that precise.

**Definition 2.4.** The *Gauss norm* on $T_{n,K}$ is the $K$-algebra norm $\|\cdot\|$, defined by

$$\left\| \sum_{i \in \mathbb{Z}_{\geq 0}^n} c_i x^i \right\| = \sup_i |c_i|.$$

One immediately checks that the Gauss norm satisfies the requirements to be a $K$-algebra norm. Furthermore one verifies that

$$\{f \in T_{n,K} \mid \|f\| \leq 1\} = \mathcal{O}_K\langle x_1, \ldots, x_n\rangle.$$

We will use the following often:

**Lemma 2.5.** *For any nonzero $f \in T_{n,K}$, there is an $a \in K^\times$ such that $\|af\| = 1$.*

*Proof.* Suppose that $f = \sum_i c_i x^i \in T_{n,K}$. Then by definition $(c_i) \to 0$, and so we see that the supremum in definition 2.4 is really achieved. That is, there exists $i$ such that $\|f\| = c_i$. Since $f \neq 0$ by assumption, so too does not $c_i$. Then put $a = c_i^{-1}$. □

We have another useful property of the Gauss norm.

**Proposition 2.6.** *The Gauss norm is multiplicative. That is, $\|fg\| = \|f\| \cdot \|g\|$ for all $f, g \in T_{n,K}$.*

*Proof.* Due to $\|\cdot\|$ being a $K$-algebra norm we know that $\|fg\| \leq \|f\| \cdot \|g\|$. By lemma 2.5, we can assume without any loss of generality that $\|f\| = \|g\| = 1$. Then we see $f, g \in \mathcal{O}_K\langle x_1, \ldots, x_n\rangle$. Recall we had the reduction map

$$\mathrm{red}\colon \mathcal{O}_K\langle x_1, \ldots, x_n\rangle \to \kappa[x_1, \ldots, x_n].$$

The kernel of this map are those $h \in \mathcal{O}_K\langle x_1, \ldots, x_n\rangle$ who have $\|h\| < 1$. Your favorite isomorphism theorem then yields

$$\frac{\mathcal{O}_K\langle x_1, \ldots, x_n\rangle}{\ker(\mathrm{red})} \cong \kappa[x_1, \ldots, x_n].$$

Now $\kappa[x_1, \ldots, x_n]$ is an integral domain, whence $\ker(\mathrm{red})$ is prime. Therefore, since neither $f$ nor $g$ lie in $\ker(\mathrm{red})$, it follows $fg \notin \ker(\mathrm{red})$. Thus $\|fg\| = 1$. □

Recall that a *Banach $K$-algebra* is a $K$-algebra together with a norm with respect to which it is complete. The Tate algebra $T_{n,K}$ equipped with the Gauss norm forms a Banach $K$-algebra (cf. [HS22], prop. 41).

**Definition 2.7.** Let $f \in T_{n,K}$. For $f_i \in T_{n-1,K}$, write

$$f = \sum_{i \in \mathbb{Z}_{\geq 0}} f_i x_n^i.$$

Then, if for some $k \in \mathbb{Z}_{\geq 0}$

1. The element $f_k$ is a unit in $T_{n-1,K}$ and
2. for all $j > k$, $\|f_j\| < \|f_k\|$ and $\|f\| = \|f_k\|$,

$f$ is said to be *distinguished of order $k$*.

As an example, take $f$ nonzero in $T_{1,K}$. Then for some $k$ we have $\|f\| = |c_k|$ and hence $f$ is distinguished of order $k$. We will apply these ideas to prove the following, which is sort of like a Euclidean division algorithm.

**Theorem 2.8.** *(Weierstrass Division Theorem) Let $g \in T_{n,K}$ be distinguished of order $k$. Then for $f \in T_{n,K}$, there exist unique $q \in T_{n,K}$, $r \in T_{n-1,K}[x_n]$ with $\deg r < k$ such that*

$$f = g \cdot q + r,$$

*with $\|f\| = \max\{\|q\| \cdot \|g\|, \|r\|\}$.*

Before proving it, let us note an immediate corollary:

**Corollary 2.9.** *The Tate algebra in one variable, $T_{1,K} = K\langle x\rangle$ is a Euclidean domain. In particular it is a principal ideal domain.*

*Proof.* We saw before that every $f \in T_{1,K}$ is distinguished of some order. Thus we can apply Weierstrass division to every $f \in T_{1,K}$. □

To prove the Weierstrass division theorem we need a couple of lemmas.

**Lemma 2.10.** *Let $f, g \in T_{n,K}$ with $g$ distinguished of order $k$. Suppose that there exist $q \in T_{n,K}$, $r \in T_{n-1,K}[x_n]$ with $\deg r < k$ such that*

$$f = g \cdot q + r.$$

*Then $\|f\| = \max\{\|q\| \cdot \|g\|, \|r\|\}$.*

*Proof.* We can assume that both $q$ and $r$ are nonzero, otherwise the claim is vacuously true. Then once again lemma 2.5 allows us to assume with no generality lost that $\|g\| = \max\{\|q\| \cdot \|g\|, \|r\|\} = 1$. By the property of $K$-algebra norms we have

$$\|f\| \leq \max\{\|q\| \cdot \|g\|, \|r\|\}. \qquad (*)$$

Suppose that it is a strict inequality. Then

$$\mathrm{red}(f) = \mathrm{red}(g \cdot q + r) = 0.$$

But by $(*)$ either $\|q\| = 1$ or $\|r\| = 1$ and so either $\mathrm{red}(q)$ or $\mathrm{red}(r)$ is nonzero. This however contradicts the Euclidean division in $\kappa[x_1, \ldots, x_n]$. □

**Lemma 2.11.** *Let $g \in T_{n,K}$ be distinguished of order $k$ with $\|g\| = 1$. Write*

$$g = \sum_{i \in \mathbb{Z}_{\geq 0}} g_i x_n^i$$

*for $g_i \in T_{n-1,K}$. Then put*

$$g = \underbrace{\sum_{i=0}^{k} g_i x_n^i}_{g'} + \underbrace{\sum_{i=k+1}^{\infty} g_i x_n^i}_{g''},$$

*such that $g'$ is distinguished of order $k$ and $\|g'\| = 1$. Let $\varepsilon = \|g''\| < 1$. For $f \in T_{n,K}$, there are elements $q, f_1 \in T_{n,K}$ and $r \in T_{n-1,K}[x_n]$ with $\deg r < k$ such that*

1. *$f = q \cdot g + r + f_1$,*
2. *$\|f_1\| \leq \varepsilon \|f\|$,*
3. *and both $\|q\|$ and $\|r\|$ are less than or equal to $\|f\|$.*

*Proof.* See ([HS22], lemma 47). □

*Proof of Weierstrass Division.* Clearly from lemma 2.10, if we are able to produce the $q$ and $r$ as described in the theorem statement then it will hold that $\|f\| = \max\{\|q\| \cdot \|g\|, \|r\|\}$. Now assume that $\|g\| = 1$ and, as in lemma 2.11, put $g = g' + g''$. Let $\varepsilon = \|g''\|$. We now apply lemma 2.11. Setting $f = f_0$, for each $i \in \mathbb{Z}_{\geq 0}$ we find $f_i, q_i \in T_{n,K}$ and $r_i \in T_{n-1,K}[x_n]$ such that

$$f_i = q_i \cdot g + r_i + f_{i+1}.$$

Then by the lemma we have $\|q_i\|, \|r_i\| \leq \varepsilon^i \|f\|$ and $\|f_{i+1}\| \leq \varepsilon^{i+1} \|f\|$. Let $q = \sum_{i=0}^{\infty} q_i$, $r = \sum_{i=0}^{\infty} r_i$. It now is an easy check that

$$f = g \cdot (q_0 + q_1 + \cdots) + (r_0 + r_1 + \cdots) = g \cdot q + r.$$

What is left is to check the unicity of $q$ and $r$ as demanded by the statement of the theorem. Suppose we have $q, r$ such that $f = g \cdot q + r$ and there exist different $q', r'$ satisfying $f = g \cdot q' + r'$. This means

$$0 = g \cdot (q - q') + (r - r').$$

Well then lemma 2.10 says that

$$0 = \|0\| = \max\{\|g\| \cdot \|(q - q')\|, \|(r - r')\|\}.$$

Now $g$ is distinguished of order $k$ by assumption and so is nonzero. Thus

$$\|(q - q')\| = \|(r - r')\| = 0,$$

whence $q = q'$ and $r = r'$. □

We now will state without proof the Weierstrass preparation theorem, but remark that it is in fact an equivalent result to the Weierstrass division theorem: One can be deduced from the other and *vice versa.*

**Theorem 2.12.** *(Weierstrass Preparation Theorem) Let $g \in T_{n,K}$ be distinguished of order $k$. Then there is a unique monic polynomial $\omega \in T_{n-1,K}[x_n]$ and unit $e \in T_{n,K}$ such that*

$$g = e\omega.$$

*Furthermore $\|\omega\| = 1$ and $\omega$ is distinguished of order $k$.*

One point of concern for the reader could be that the hypotheses of many of these results demand that the elements in $T_{n,K}$ be distinguished. This lack of generality would prove troublesome, however it is the case that, after a change of variables if necessary, all elements of $T_{n,K}$ are distinguished of some order (cf. [HS22], lemma 51). We next will demonstrate how these results are used by proving that $T_{n,K}$ is Noetherian.

**Proposition 2.13.** *The algebra $T_{n,K}$ is Noetherian. That is, all ideals are finitely generated.*

*Proof.* Note that for $n = 0$ then it is trivially true. Indeed any field is Noetherian. Assume that the algebra $T_{n-1,K}$ is Noetherian, and let $\mathfrak{a} \subseteq T_{n,K}$ be any ideal. Choose $g \in \mathfrak{a}$, which, after a change of variables if necessary, we assume to be distinguished of order $k$. We now consider $T_{n,K}/(g)$ and apply theorem 2.8 to see that it is generated by $\{1, x_n, \ldots, x_n^k\}$ as a $T_{n-1,K}$-module. Hence $T_{n,K}/(g)$ is a Noetherian $T_{n-1,K}$-module sincee $T_{n-1,K}$ is assumed to be Noetherian. Since $\mathfrak{a}$ is an ideal above $(g)$, it follows that it is finitely generated. □

We now will state and collect in a proposition other properties which are possessed by the algebra $T_{n,K}$, while directing the reader to ([Bos14], §2.2) for detailed proofs.

**Proposition 2.14.** *The following hold:*

1. *The algebra $T_{n,K}$ is a unique factorization domain and in particular is integrally closed in its field of fractions.*

2. *(Noether Normalization) For any proper ideal $\mathfrak{a} \subset T_{n,K}$, there is an injective $K$-algebra homomorphism $T_{d,K} \hookrightarrow T_{n,K}$ such that the composition $T_{d,K} \hookrightarrow T_{n,K} \to T_{n,K}/\mathfrak{a}$ is a finite injection. The integer $d$ is equal to the Krull dimension of $T_{n,K}/\mathfrak{a}$.*

3. *The algebra $T_{n,K}$ is Jacobson. That is to say for any ideal $\mathfrak{a} \subseteq T_{n,K}$, its nilradical is equal to the intersection of all maximal ideals $\mathfrak{m} \in \operatorname{Spec_m} T_{n,K}$ that contain $\mathfrak{a}$.*

4. *Every maximal ideal $\mathfrak{m} \in \operatorname{Spec_m} T_{n,K}$ is of height $n$ and can be generated by $n$ elements. In particular, the Krull dimension of $T_{n,K}$ is $n$.*

**Corollary 2.15** (Corollary to (2))**.** *Let $\mathfrak{m} \in \operatorname{Spec_m} T_{n,K}$. Then $T_{n,K}/\mathfrak{m}$ is a finite extension of $K$.*

*Proof.* Noether normalization says that there is a finite injective $K$-algebra homomorphism $T_{d,K} \hookrightarrow T_{n,K}/\mathfrak{m}$ for some $d \in \mathbb{Z}_{\geq 0}$. But $T_{n,K}/\mathfrak{m}$ is a field and so $T_{d,K}$ must too be a field. Thus $d = 0$ and $T_{d,K} = K$. Therefore $T_{n,K}/\mathfrak{m}$ is finite over $K$. □

This shows that the Tate algebra $T_{n,K}$ has many of the properties held by the polynomial ring $k[x_1, \ldots, x_n]$ for a field $k$ which allow us to work with varieties over $k$. Equipped with this knowledge, we continue to develop our theory inspired by that of varieties.

## 3 Affinoid Algebras and Affinoid Spaces

We again relate to varieties. An affine variety over a field $k$ is cut out by the zero loci of polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$. Then the points of the variety are precisely the maximal ideals

$$\mathfrak{m} \in \operatorname{Spec}_{\mathrm{m}} \left( \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_m)} \right)$$

of the coordinate ring $k[x_1, \dots, x_n]/(f_1, \dots, f_m)$, and elements of the coordinate ring are thought of as functions on the affine variety. The situation is nearly identical in the rigid analytic setting.

Lemma 2.1 told us that we can view an element $f \in T_{n,K}$ as a function $f \colon \mathbb{B}_1^n(0, K^a) \to K^a$. Consider an ideal $\mathfrak{a} \subseteq T_{n,K}$. We can define the set

$$V(\mathfrak{a}) = \{x \in \mathbb{B}_1^n(0, K^a) \mid f(x) = 0 \text{ for all } f \in \mathfrak{a}\}$$

and restrict functions on $\mathbb{B}_1^n(0, K^a)$ to $V(\mathfrak{a})$. This restriction homomorphism clearly vanishes on $\mathfrak{a}$ and in this way we can view $T_{n,K}/\mathfrak{a}$ as an algebra of functions on $V(\mathfrak{a})$. It is algebras of these type which we study in this section.

**Definition 3.1.** A $K$-algebra $A$ is an *affinoid $K$-algebra* if there exists a $K$-algebra surjection $T_{n,K} \to A$ for some $n$.

Thus we see that the affinoid $K$-algebras are quotients of $T_{n,K}$ for some $n$. A morphism of affinoid $K$-algebras is nothing but a $K$-algebra morphism. Thus affinoid $K$-algebras form a full subcategory of $\mathrm{Alg}_K$, which we will denote by $\mathrm{AffAlg}_K$. The affinoid algebras are also quite well behaved:

**Proposition 3.2.** *Let $A$ be an affinoid $K$-algebra. Then,*

1. *$A$ is Noetherian,*
2. *$A$ is Jacobson, and*
3. *$A$ satisfies Noetherian normalization. That is, there is a finite injection $T_{d,K} \hookrightarrow A$ for some $d$.*

We define two norms (really one is a seminorm) on an affinoid algebra $A$.

**Definition 3.3.** Let $\alpha \colon T_{n,K} \to A$ be a surjection onto an affinoid $K$-algebra $A$. We define the *residue norm*, $|\cdot|_\alpha$, on $A$ as

$$|f|_\alpha = \inf_{f' \in \alpha^{-1}(f)} \|f'\|.$$

From the definition of the residue norm alone it seems that the choice of surjection $\alpha$ affects the function $|\cdot|_\alpha$. However, the topology induced by $|\cdot|_\alpha$ is independent of $\alpha$.

**Proposition 3.4.** *Let $\alpha \colon T_{n,K} \to A$ and $\beta \colon T_{m,K} \to B$ be two surjections of a Tate algebra onto an affinoid algebra. Then with respect to the topologies on $A$ and $B$ induced by $\alpha$ and $\beta$ respectively, any affinoid $K$-algebra homomorphism $\varphi \colon A \to B$ is continuous. In particular, taking $\varphi = \mathrm{id}_A$, we see any two residue norms on an affinoid algebra $A$ are equivalent.*

*Proof.* See ([Bos14], Ch. 3 §1, lemma 19, prop. 20). □

We topologize an affinoid algebra $A$ with the topology induced by a residue norm.

Recall that if $X$ is an affine variety with coordinate ring $\mathcal{O}(X)$, then we define the evaluation of an element $f \in \mathcal{O}(X)$ at a point $\mathfrak{m} \in \operatorname{Spec}_{\mathrm{m}} \mathcal{O}(X) = X$ as

$$f(\mathfrak{m}) = (\mathcal{O}(X) \twoheadrightarrow \mathcal{O}(X)/\mathfrak{m})(f).$$

We do something similar for elements of an affinoid algebra $A$. For $\mathfrak{m} \in \operatorname{Spec}_m A$, put

$$f(\mathfrak{m}) = (A \twoheadrightarrow A/\mathfrak{m})(f).$$

**Definition 3.5.** Let $f \in A$. The *supremum seminorm*, $\|\cdot\|_{\sup}$, is defined as

$$\|f\|_{\sup} = \sup_{\mathfrak{m} \in \operatorname{Spec}_m A} |f(\mathfrak{m})|,$$

where by $|f(\mathfrak{m})|$ above we understand the extended absolute value from $K$ to its finite extension $A/\mathfrak{m}$.

*Remark* 1. Note that the supremum seminorm is just a seminorm unless $A$ is reduced, where it then is really a norm.

Consider an open subring $A_0 \subseteq A$ of an affinoid algebra $A$. Seeing as $A_0$ is open, it necessarily contains a power of the uniformizer $\pi$. Thus we can talk about the $\pi$-adic topology on the subring $A_0$. This motivates the following definition.

**Definition 3.6.** A *ring of definition* is an open subring $A_0 \subseteq A$ of an affinoid algebra $A$ such that the subspace topology on $A_0$ coincides with the $\pi$-adic topology.

*Remark* 2. A ring of definition is automatically $\pi$-adically complete: Any open subgroup of a topological group is closed.

The following gives us a way to produce rings of definition.

**Lemma 3.7.** *Let $A$ be an affinoid algebra and $\alpha\colon T_{n,K} \to A$ a surjection. Then, the subring*

$$A_0 = \{f \in A \mid |f|_\alpha \leq 1\}$$

*is a ring of definition in $A$.*

*Proof.* The definition of $A_0$ gives that $\alpha$ restricts to a surjection

$$\alpha\colon \mathcal{O}_K\langle x_1, \dots, x_n\rangle \to A_0.$$

As $\alpha$ is a quotient map, this means that $A_0 \subseteq A$ is open. Then, since the subspace topology on $\mathcal{O}_K\langle x_1, \dots, x_n\rangle$ is the $\pi$-adic topology, so too is the topology on its quotient $A_0$. □

We will refer to the ring $A_0$ as in lemma 3.7 as the ring of definition *associated to* or *arising from* the surjection $\alpha$.

**Lemma 3.8.** *For any ring of definition $A_0$ of an affinoid $K$-algebra $A$, it holds that $A_0[1/\pi] = A$.*

*Proof.* Let $f \in A$. Then the sequence $(\pi^n f)$ converges to 0. Hence for $n$ large enough $\pi^n f \in A_0$. □

**Definition 3.9.** A set $S \subseteq A$ is *bounded* if for any open neighborhood $U \subseteq A$ of 0, there exists an integer $k \geq 0$ such that $\pi^k S \subseteq U$. An element $f \in A$ is *power bounded* if the set $\{f^n \mid n \in \mathbb{Z}_{\geq 0}\}$ is power bounded. We put

$$A^{\bigcirc} = \{f \in A \mid f \text{ is power bounded}\}.$$

**Lemma 3.10.** *Any morphism $A \to B$ of affinoid algebras restricts to a map $A^{\bigcirc} \to B^{\bigcirc}$ on power bounded elements.*

*Proof.* Let $\varphi\colon A \to B$ be a map of affinoid $K$-algebras. Recall (prop. 3.4) that $\varphi$ is then necessarily continuous. Let $f \in A^{\bigcirc}$ and let $V \subseteq B$ be an open neighborhood of 0. Then $\varphi^{-1}(V)$ is an open neighborhood of 0 in $A$ so there exists an integer $k \geq 0$ such that $\{\pi^k f^n\}_{n \in \mathbb{Z}_{\geq 0}} \subseteq \varphi^{-1}(V)$. Hence $\varphi(\pi^k f^n) = \pi^k \varphi(f)^n \in V$ for all $n \in \mathbb{Z}_{\geq 0}$. Whence $\varphi(f) \in B^{\bigcirc}$. □

**Proposition 3.11.** *Let $f_1, \ldots, f_n \in A$. The following are equivalent:*

1. *There is a homomorphism $T_{n,K} \to A$ which maps $x_i$ to $f_i$.*
2. *For all $i$, the element $f_i$ is power bounded.*

*Proof.* The fact that (1) implies (2) follows from lemma 3.10. Conversely, suppose that elements $f_1, \ldots, f_n \in A$ are power bounded. Write

$$F = \sum_{i \in \mathbb{Z}^n_{\geq 0}} c_i x^i \in T_{n,K}.$$

Then by definition, the sequences $(c_i f^i) \to 0$ and thus $F \mapsto F(f_1, \ldots, f_n)$ is a well-defined continuous $K$-algebra homomorphism. □

**Corollary 3.12.** *For any $f_1, \ldots, f_m \in A^\circ$, the subring $A[f_1, \ldots, f_m] \subseteq A$ is a ring of definition associated to the surjection $T_{n+m,K} \to A$ given by $x_{n+i} \mapsto f_i$.*

We apply these ideas to show the category $\mathrm{AffAlg}_K$ has a symmetric monoidal structure.

**Proposition 3.13.** *For any diagram $B \xleftarrow{\varphi_1} A \xrightarrow{\varphi_2} C$ in $\mathrm{AffAlg}_K$, the pushout, denoted $B\widehat{\otimes}_A C$, exists.*

*Proof.* We define $B\widehat{\otimes}_A C$ as follows. Let $A_0 \subseteq A$ be the ring of definition arising from some surjection $T_{n,K} \to A$. From corollary 3.12 we can find rings of definition $B_0 \subseteq B$ and $C_0 \subseteq C$ such that $\varphi_1$ and $\varphi_2$ restrict to maps $A_0 \to B_0$ and $A_0 \to C_0$, respectively. We let

$$B_0 \widehat{\otimes}_{A_0} C_0 = (\widehat{B_0 \otimes_{A_0} C_0})_{(\pi)},$$

the completion of $B_0 \otimes_{A_0} C_0$ with respect to the $\pi$-adic topology. Then, $B\widehat{\otimes}_A C$ is defined as

$$B\widehat{\otimes}_A C = (B_0 \widehat{\otimes}_{A_0} C_0)\left[\frac{1}{\pi}\right].$$

We first show that $B\widehat{\otimes}_A C$ satisfies the universal property of pushouts, then that it is actually in $\mathrm{AffAlg}_K$. Let $D$ be an affinoid algebra making

$$\begin{array}{ccc} A & \xrightarrow{\varphi_1} & B \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

commute. As before we can find a ring of definition $D_0 \subseteq D$ such that $B \to D$ and $C \to D$ restrict to $B_0 \to D_0$ and $C_0 \to D_0$, respectively. From this we get a map $B_0 \otimes_{A_0} C_0 \to D_0$. When we complete and invert $\pi$ we then get a map $B\widehat{\otimes}_A C \to D$. Then unicity of such a map follows from the density of $B_0 \otimes_{A_0} C_0$ in $B_0 \widehat{\otimes}_{A_0} C_0$.

To show that $B\widehat{\otimes}_A C$ is affinoid, we will only sketch the proof, and direct the reader to ([Bos14], Appendix 2) for the argument in its full detail. First, we observe that

$$T_{n,K} \widehat{\otimes}_K T_{m,K} \cong T_{n+m,K}.$$

Indeed, we unpack what is meant by $T_{n,K} \widehat{\otimes}_K T_{m,K}$. We first consider the usual tensor product of the respective rings of definition. These rings of definition are just the restricted power series with coefficients in $\mathcal{O}_K$:

$$T_{n,K} \widehat{\otimes}_K T_{m,K} = (\mathcal{O}_K\langle x_1, \ldots x_n\rangle \widehat{\otimes_{\mathcal{O}_K}} \mathcal{O}_K\langle x_1, \ldots, x_m\rangle)_{(\pi)}\left[\frac{1}{\pi}\right].$$

Inside of the parentheses is just $\mathcal{O}_K\langle x_1, \ldots, x_{n+m}\rangle$. Then completing (it is already complete) and inverting $\pi$ clearly yields $T_{n+m,K}$. Using this, one then shows that, for ideals $\mathfrak{a} \subseteq T_{n,K}$, $\mathfrak{b} \subseteq T_{m,K}$, one has

$$T_{n,K}/\mathfrak{a}\widehat{\otimes}_K T_{m,K}/\mathfrak{b} \cong T_{n+m,K}/(\mathfrak{a}, \mathfrak{b}),$$

whence the algebra $B\widehat{\otimes}_K C$ for affinoid algebras $B$ and $C$ is affinoid. Finally, one can show that there is always a surjection

$$B\widehat{\otimes}_K C \to B\widehat{\otimes}_A C$$

for the affinoid $K$-algebra $A$, and thus $B\widehat{\otimes}_A C$ is affinoid. □

**Definition 3.14.** The amalgamated sum as in proposition 3.13 is called the *completed tensor product*.

We now shift our attention from affinoid algebras to *affinoid spaces*. To an affinoid algebra $A$, we associate the corresponding affinoid space $\operatorname{Spec_m} A$, which is the set of the maximal ideals of $A$ together with its $K$-algebra of functions, $A$.

**Proposition 3.15.** *A morphism of affinoid $K$-algebras $\varphi\colon A \to B$ induces a map $\psi\colon \operatorname{Spec_m} B \to \operatorname{Spec_m} A$, given by $\psi(\mathfrak{m}) = \varphi^{-1}(\mathfrak{m})$.*

*Proof.* Indeed $\varphi^{-1}(\mathfrak{m})$ is maximal: We have a chain of injections

$$K \hookrightarrow A/\varphi^{-1}(\mathfrak{m}) \hookrightarrow B/\mathfrak{m}.$$

Since $B/\mathfrak{m}$ is finite over $K$, it follows that the ring $A/\varphi^{-1}(\mathfrak{m})$ must be a field. □

In this way we see the category of affinoid spaces over $K$, denoted $\mathrm{AffSp}_K$, is opposite to $\mathrm{AffAlg}_K$. Our goal is to be able to do analytic geometry on an affinoid space. If by "do analytic geometry" we mean have a well behaved sheaf of analytic functions on our space, then that demands that we topologize affinoid spaces and define a sheaf on them. There are a couple of immediate candidates for how we should go about this.

On one hand, seeing as an affinoid space $X$ is defined as $X = \operatorname{Spec_m} A$ for some affinoid algebra $A$, one has $\operatorname{Spec_m} A \subseteq \operatorname{Spec} A$, whence we could endow $X$ with the topology inherited from the Zariski topology on $\operatorname{Spec} A$. Perhaps unsurprisingly, the Zariski topology is far too coarse to beget any sort of useful analytic theory.

On the other hand, proposition 2.3 suggests that $X = \operatorname{Spec_m} A$ can be topologized by inheriting that of $\mathbb{B}_1^n(0, K^a)$ (we call this the *canonical topology*). The totally disconnected topology of nonarchimedean local fields however causes problems here too. This is exemplified by considering the open cover of $\mathbb{B}_1(0, K^a)$:

$$\mathbb{B}_1(0, K^a) = \bigcup_{n\geq 1} \{x \in K^a \mid |x| \leq |\pi|^{1/n}\} \sqcup \{x \in K^a \mid |x| = 1\}.$$

We will call the first term in the disjoint union above $\mathbb{D}_1(K^a)$ (the open disk) and the second term $\mathbb{T}_1(K^a)$ (the analytic 1-torus). This decomposition suggests that for a sheaf, $\mathcal{O}$, of analytic functions on $\mathbb{B}_1(0, K^a)$ we should have

$$\mathcal{O}(\mathbb{B}_1(0, K^a)) = \mathcal{O}(\mathbb{D}_1(K^a)) \times \mathcal{O}(\mathbb{T}_1(K^a)).$$

This certainly does not describe the behavior of analytic functions that one should expect. Instead one would want that the behavior of an analytic function on a closed unit disk should be determined by its behavior on the interior.

Thus we see that the Zariski and canonical topologies exist on opposite extremes of a spectrum: The Zariski topology is far too coarse and the canonical topology is far too fine. Something in between ought to do the trick!

**Definition 3.16.** Let $X = \operatorname{Spec_m} A$ be an affinoid space. A subset $U \subseteq X$ is an *affinoid subdomain* if there exists an affinoid $K$-space map $\iota\colon X' = \operatorname{Spec_m} A' \to X$ such that $\iota(X') \subseteq U$ and such that the following universal property is satisfied: For any affinoid $K$-space map $f\colon Y \to X$ with $f(Y) \subseteq U$, there is a unique map $f'\colon Y \to X'$ such that $f = \iota \circ f'$. In other words, the diagram commutes:

$$\begin{array}{ccccc} X' & \xrightarrow{\iota} & U & \xrightarrow{\text{include}} & X \\ {\scriptstyle f'}\uparrow & \nearrow_{f} & & & \\ Y & & & & \end{array}$$

**Lemma 3.17.** *Suppose $U \subseteq X = \operatorname{Spec_m} A$ is an affinoid subdomain with universal morphism*

$$\iota\colon \operatorname{Spec_m} B \to U.$$

*Then $\iota$ is a bijection.*

*Proof.* See ([HS22], lemma 112). □

In this way we can identify $U$ with the affinoid space $\operatorname{Spec_m} B$, and thus the affinoid algebra $B$ with "functions on $U$". Henceforth we will say $U$ is *represented* by the map $\iota$, by the affinoid space $\operatorname{Spec_m} B$, or by the affinoid algebra $B$, all interchangeably.

*Remark* 3. One shows that affinoid subdomains are open in the canonical topology (cf. [Bos14], Ch. 3 §3).

This definition is a tad abstract. Let us consider an example. Let $A$ be an affinoid algebra with affinoid space $X = \operatorname{Spec_m} A$. For $f_1, \dots, f_m \in A$, define

$$X(f_1, \dots, f_m) = \{x \in X \mid |f_i(x)| \leq 1\}.$$

Then $X(f_1, \dots, f_m)$ (called a *Weierstrass domain*) is an affinoid subdomain represented by

$$\iota\colon X' = \operatorname{Spec_m}(A\langle f_1, \dots, f_m\rangle) \to X,$$

where $A\langle f_1, \dots, f_m\rangle = A\langle t_1, \dots, t_m\rangle/(t_i - f_i)$. Suppose that $\varphi\colon Y = \operatorname{Spec_m} B \to X$ is a affinoid space map factoring setwise through $X(f_1, \dots, f_m)$. Denote by $\varphi^*$ the corresponding map on affinoid algebras. Then one has

$$|\varphi^*(f_i)(y)| = |f_i(\varphi(y))| \leq 1$$

for all $y \in \operatorname{Spec_m} B$. Thus $\|\varphi^*(f_i)\|_{\sup} \leq 1$, implying that $\varphi^*(f_i)$ is power bounded (cf. [HS22], prop. 98). Now (cf. [Tia], corollary 1.4.14) there exists a map

$$\psi^*\colon A\langle t_1, \dots, t_m\rangle \to B$$

such that $\psi^*|_A = \varphi^*$ and $\psi^*(t_i) = \varphi^*(f_i)$. In other words, $\psi^*$ factors through the quotient $A\langle t_1, \dots, t_m\rangle/(t_i - f_i)$, whence we have $\psi\colon \operatorname{Spec_m} B \to X$ factoring through $X'$. Therefore $X(f_1, \dots, f_m)$ is an affinoid subdomain.

**Definition 3.18.** Let $f_0, \dots, f_r \in A$ be such that they generate the unit ideal. We define the subset of $X = \operatorname{Spec_m} A$

$$X\left(\frac{f_1, \dots, f_r}{f_0}\right) = \{x \in X \mid |f_i(x)| \leq |f_0(x)|, i = 1, \dots, r\}.$$

Any subset of this form is called a *rational subdomain* of $X$.

*Remark* 4. The condition that the $f_i$ generate the unit ideal amounts to saying that they do not share a common zero. In particular, this implies that for all $x \in X(f_1, \ldots, f_r/f_0)$, $f_0(x) \neq 0$. If $f_0(x) = 0$ for some $x \in X(f_1, \ldots, f_r/f_0)$, then $|f_i(x)| \leq |f_0(x)| = 0$ means that all the $f_i$ vanish at $x$ too. This contradicts the fact that they do not have a common zero.

**Proposition 3.19.** *All rational subdomains are affinoid subdomains.*

*Proof.* The rational subdomain $X(f_1, \ldots, f_r/f_0)$ is represented by

$$\iota\colon \operatorname{Spec_m}\left(\frac{A\langle t_1, \ldots, t_r\rangle}{(f_i - f_0 t_i)}\right) \to X.$$

For full details see ([Tia], example 1.5.12). It is similar to the Weierstrass domain situation. □

**Definition 3.20.** Let $f \in A$. Then we write $X(f)$ for the rational subdomain $X(f/1)$ and $X(f^{-1})$ for the rational subdomain $X(1/f)$. For $X = \operatorname{Spec_m} A$, we have $X = X(f) \cup X(f^{-1})$. We call this a *simple Laurent covering.* A *Laurent domain* is an affinoid subdomain of the form

$$X(f_1) \cap \cdots \cap X(f_n) \cap X(g_1^{-1}) \cap \cdots \cap X(g_m^{-1}).$$

The key part of the discussion of affinoid and rational subdomains is the following theorem:

**Theorem 3.21** (Gerritzen-Grauert, 1969)**.** *Every affinoid subdomain of an affinoid space* $X = \operatorname{Spec_m} A$ *is a finite union of rational subdomains.*

The proof is quite technical. The original paper of Gerritzen and Grauert ([GG69]) is available in German. A proof is reproduced in ([BGR12], §7.3.5) in English. An alternative proof was given in 2005 by Michael Temkin in ([Tem05]).

The upshot of this is that we know the affinoid algebras which represent the rational subdomains, and hence the functions on them. This is very useful for developing a well behaved sheaf on our space. The theorem of Gerritzen and Grauert then tells us that all affinoid subdomains are built up by these rational subdomains. This is similar to what we do in algebraic geometry. We consider *distinguished opens* (complements of the vanishing locus of a single polynomial), on which we understand well the functions defined, then build up the topology from there.

## 4 Tate's Acyclicity Theorem

Let $X = \operatorname{Spec_m} A$ be an affinoid space. Define $\mathscr{T}$ to be the category of affinoid subdomains of $X$ with inclusions as morphisms. We can then get the presheaf $\mathcal{O}_X$ on $\mathscr{T}$ by associating to an affinoid subdomain the algebra representing it.

**Definition 4.1.** Let $\mathscr{U} = \{U_i \mid i \in I\}$ be a covering of $X$ by affinoid subdomains (i.e. $X = \bigcup_{i\in I} U_i$). Then, with respect to a fixed ordering of $I$, we define the *Čech complex*, denoted $\check{\mathscr{C}}(\mathscr{U}, \mathcal{O}_X)$, to be the complex

$$\prod_i \mathcal{O}_X(U_i) \to \prod_{i<j} \mathcal{O}_X(U_i \cap U_j) \to \prod_{i<j<k} \mathcal{O}_X(U_i \cap U_j \cap U_k) \to \cdots$$

The cohomology of this complex is written $\check{H}^*(\mathscr{U}, \mathcal{O}_X)$.

**Definition 4.2.** A covering $\mathscr{U}$ of $X$ is said to be $\mathcal{O}_X$*-acyclic* if $\mathcal{O}_X(X) \to \check{\mathscr{C}}(\mathscr{U}, \mathcal{O}_X)$ is a resolution. Equivalently, if $\check{H}^0(\mathscr{U}, \mathcal{O}_X) = A$ while $\check{H}^i(\mathscr{U}, \mathcal{O}_X)$ vanishes in all higher degrees. In particular, this means that $\mathcal{O}_X$ satisfies the sheaf condition on $\mathscr{U}$.

**Theorem 4.3** (Tate, 1971)**.** *Any finite covering of an affinoid space $X$ by affinoid subdomains is $\mathcal{O}_X$-acyclic.*

Note that the theorem is saying that the presheaf $\mathcal{O}_X$ acts as a sheaf on finite covers by affinoid subdomains. A full proof for theorem 4.3 can be found in Tate's original paper, ([Tat71], theorem 8.2), as well as in ([BGR12], Ch. 8 §2). For the sake of completeness, in this paper we include the proof (following closely that produced in ([Bos14], Ch. 4 §3)) of a slightly weaker statement:

**Theorem 4.4.** *Let $X$ be an affinoid space. The presheaf $\mathcal{O}_X$ is a $\mathscr{U}$-sheaf for all finite coverings $\mathscr{U} = (U_i)_{i\in I}$ of $X$ by affinoid subdomains $U_i \subseteq X$.*

*Proof.* The proof is somewhat lenghty, and the entirety of what remains of this section is dedicated to it. Let $\mathscr{U} = (U_i)_{i\in I}$ and $\mathscr{V} = (V_j)_{j\in J}$ be two coverings of $X$. Then we say that $\mathscr{V}$ is a *refinement* of $\mathscr{U}$ if there exists $\tau\colon J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for all $j \in J$. In the following discussion, we let $\mathscr{F}$ be any presheaf on $X$. We will need a series of lemmas.

**Lemma 4.5.** *Let $\mathscr{U} = (U_i)_{i\in I}$ and $\mathscr{V} = (V_j)_{j\in J}$ be coverings of $X$ by affinoid subdomains, where $\mathscr{V}$ is a refinement of $\mathscr{U}$. Then if $\mathscr{F}$ is a $\mathscr{V}$-sheaf, it is also a $\mathscr{U}$-sheaf.*

*Proof of lemma 4.5.* Recall the sheaf condition: A presheaf $\mathscr{F}$ is a $\mathscr{U}$-sheaf if the sequence

$$\mathscr{F}(X) \to \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j) \tag{$*$}$$

is exact, where the first map is given by

$$f \mapsto (f|_{U_i})_{i\in I},$$

and the second is given by

$$(f_i)_{i\in I} \mapsto \begin{cases} (f_i|_{U_i\cap U_j})_{i,j\in I} \\ (f_j|_{U_i\cap U_j})_{i,j\in I}. \end{cases}$$

We will prove that $(*)$ is exact for the covering $\mathscr{U}$. Consider the $f_i \in \mathscr{F}(U_i)$, $i \in I$ such that $f_i|_{U_i\cap U_{i'}} = f_{i'}|_{U_i\cap U_{i'}}$ for all $i, i' \in I$. Then choose $\tau\colon J \to I$ such that $V_j \subseteq U_{\tau(j)}$ and put $g_j = f_{\tau(j)}|_{V_j}$ for all $j \in J$. With this we have

$$\begin{aligned} g_j|_{V_j\cap V_{j'}} &= (f_{\tau(j)}|_{U_{\tau(j)}\cap U_{\tau(j')}})|_{V_j\cap V_{j'}} \\ &= (f_{\tau(j')}|_{U_{\tau(j)}\cap U_{\tau(j')}})|_{V_j\cap V_{j'}} = g_{j'}|_{V_j\cap V_{j'}}. \end{aligned}$$

Note that the exactness of $(*)$ is saying that the first map is a bijection onto the subset of elements who have the same image under the latter two maps. Given that $\mathscr{F}$ is a $\mathscr{V}$-sheaf, there then exists a unique $f \in \mathscr{F}(X)$ such that $f|_{V_j} = g_j$ for all $j \in J$. We claim that that $f|_{U_i} = f_i$ for all $i \in I$. To this end, fix $i \in I$. Then we have

$$(f|_{U_i})|_{U_i\cap V_j} = f|_{U_i\cap V_j} = g_j|_{U_i\cap V_j}.$$

Then,

$$f_i|_{U_i\cap V_j} = f_i|_{U_i\cap U_{\tau(j)}\cap V_j} = f_{\tau(j)}|_{U_i\cap U_{\tau(j)}\cap V_j} = g_j|_{U_i\cap V_j}.$$

Therefore $f_i|_{U_i\cap V_j} = (f|_{U_i})|_{U_i\cap V_j}$. Then since $\mathscr{F}$ is a $\mathscr{V}$-sheaf when restricted to $U_i$, it follows that $f|_{U_i} = f_i$ for all $i \in I$. As $f$ is uniquely determined by these conditions, we have that $\mathscr{F}$ is a $\mathscr{U}$-sheaf. □

**Lemma 4.6.** *Let $\mathscr{U} = (U_i)_{i\in I}$ and $\mathscr{V} = (V_j)_{j\in J}$ be coverings of $X$ by affinoid subdomains. Assume that*

1. *$\mathscr{F}$ is a $\mathscr{V}$-sheaf.*
2. *The restriction of $\mathscr{F}$ to $V_j$ is a $\mathscr{U}|_{V_j}$-sheaf for all $j \in J$.*

*Then $\mathscr{F}$ is a $\mathscr{U}$-sheaf.*

*Proof of lemma 4.6.* We again prove the sequence $(*)$ to be exact for the covering $\mathscr{U}$. As before, consider the elements $f_i \in \mathscr{F}(U_i)$ such that

$$f_i|_{U_i\cap U_{i'}} = f_{i'}|_{U_i\cap U_{i'}}$$

for all $i, i' \in I$. Then if we fix $j \in J$, we get

$$f_i|_{U_i\cap U_{i'}\cap V_j} = f_{i'}|_{U_i\cap U_{i'}\cap V_j}.$$

The assumption (2) implies the existence of a unique $g_j \in \mathscr{F}(V_j)$ such that $g_j|_{U_i\cap V_j} = f_i|_{U_i\cap V_j}$. Now fixing $j, j' \in J$, we have

$$g_j|_{U_i\cap V_j\cap V_{j'}} = f_i|_{U_i\cap V_j\cap V_{j'}} = g_{j'}|_{U_i\cap V_j\cap V_{j'}},$$

whence assumption (2) guarantees $g_j|_{V_j\cap V_{j'}} = g_{j'}|_{V_j\cap V_{j'}}$. By assumption (1), there exists a unique $g \in \mathscr{F}(X)$ such that $g|_{V_j} = g_j$ for all $j \in J$. By construction this $g$ coincides with $f_i$ when restricted to $U_i \cap V_j$ for all $i \in I$, $j \in J$. This implies by (1) that $g|_{U_i} = f_i$ for all $i$. As $g$ is uniquely determined by these conditions, it follows that $\mathscr{F}$ is a $\mathscr{U}$-sheaf. ☐

We want to apply these ideas to specific types of coverings of $X$. We will say that a *finite* covering of $X$ by affinoids is an *affinoid covering* of $X$. If we choose $f_0, \ldots, f_r \in A$ with no common zeroes, then we put

$$U_i = X\left(\frac{f_0, \ldots, f_r}{f_i}\right), \quad i = 0, \ldots, r.$$

In this way we get a finite covering $\mathscr{U} = (U_i)_{i=0,\ldots r}$ of $X$ by rational subdomains. We call this a *rational covering* of $X$ (associated to $f_0, \ldots, f_r$).

**Lemma 4.7.** *Every affinoid covering $\mathscr{U} = (U_i)_{i\in I}$ of $X$ admits a rational covering as a refinement.*

*Proof of lemma 4.7.* By theorem 3.21, we assume that $\mathscr{U}$ consists of rational subdomains. Thus we have

$$U_i = X\left(\frac{f_1^{(i)}, \ldots, f_{r_i}^{(i)}}{f_0^{(i)}}\right),$$

for, say, $i = 1, \ldots, n$ (since we have finitely many). Define

$$I = \{(\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n_{\geq 0} \mid \nu_i \leq r_i\}$$

and put

$$f_{\nu_1\ldots\nu_n} = \prod_{i=1}^n f_{\nu_i}^{(i)}.$$

Let $I'$ be the subset of $I$ such that $\nu_i = 0$ for at least one $i$. Then the functions $f_{\nu_1,\ldots,\nu_n}$ for $(\nu_1, \ldots, \nu_n) \in I'$ do not have a common zero on $X$ and therefore their associated rational covering

$\mathscr{V}$ is a covering of $X$. Suppose that $x \in X$ was a common zero for all of the functions. Then, for some $j$, $x \in U_j$. This implies that $f_0^{(j)}(x) \neq 0$, and so the product

$$\prod_{i \neq j} f_{\nu_i}^{(i)}, \ \ 0 \leq \nu_i \leq r_i,$$

must vanish at $x$. However this is a contradiction as the $f_0^{(i)}, \dots, f_{r_i}^{(i)}$ generate the unit ideal in $A$. Thus $\mathscr{V}$ is indeed a rational covering of $X$.

For the verification that $\mathscr{V}$ is indeed a refinement of $\mathscr{U}$, we direct the reader to ([Bos14], Ch. 4 §3 lemma 4). □

In order to make computations easier we want to restrict our attention to an even simpler class of coverings. For elements $f_1, \dots, f_r \in A$, the sets

$$X(f_1^{\alpha_1}, \dots, f_r^{\alpha_r}), \ \ \alpha^i = \pm 1,$$

form a finite covering of $X$ by Laurent domains (cf. definition 3.20). A covering of this form is called a *Laurent covering* (associated to $f_1, \dots, f_r$).

**Lemma 4.8.** *Let $\mathscr{U}$ be a rational covering of $X$. Then, there exists a Laurent covering $\mathscr{V}$ of $X$ such that, for each $V \in \mathscr{V}$, the covering $\mathscr{U}|_V$ is a rational covering of $V$ that is generated by units in $\mathcal{O}_X(V)$.*

*Proof of lemma 4.8.* Let $f_0, \dots, f_r \in \mathcal{O}_X(X) = A$ be functions without common zeroes on $X$ generating the rational covering $\mathscr{U}$. Then, $f_i$ is invertible on $U_i = X(f_0, \dots, f_r/f_i)$, and as its inverse attains a maximum on $U_i$, we can find $c \in K^\times$ such that

$$|c|^{-1} < \inf_{x \in X}(\max_{i=0,\dots,r} |f_i(x)|).$$

Let $\mathscr{V}$ be the Laurent covering of $x$ generated by $cf_0, \dots, cf_r$. Then $\mathscr{V}$ is as desired. Consider

$$V = X((cf_0)^{\alpha_0}, \dots, (cf_r)^{\alpha_r}) \in \mathscr{V},$$

with $\alpha_i = \pm 1$. Reordering if necessary, assume that $\alpha_0 = \dots = \alpha_s = 1$ and $\alpha_{s+1} = \dots = \alpha_r = -1$. Then we have

$$X\left(\frac{f_0, \dots, f_r}{f_i}\right) \cap V = \emptyset$$

for $i = 0, \dots, s$ since

$$\max_{i=0,\dots,s} |f_i(x)| \leq |c|^{-1} < \max_{i=0,\dots,r} |f_i(x)|$$

for all $x \in V$. From this,

$$\max_{i=0,\dots,r} |f_i(x)| = \max_{i=s+1,\dots,r} |f_i(x)|,$$

for all $x \in V$, whence $\mathscr{U}|_V$ is the rational covering of $V$ generated by $f_{s+1}|_V, \dots, f_r|_V$. These are all units in $\mathcal{O}_X(X)$ by construction. □

**Lemma 4.9.** *Let $\mathscr{U}$ be a rational covering of $X$ generated by units $f_0, \dots, f_r \in \mathcal{O}_X(X)$. Then there exists a Laurent covering $\mathscr{V}$ of $X$ that is a refinement of $\mathscr{U}$.*

*Proof of lemma 4.9.* We define $\mathscr{V}$ to be the Laurent covering generated by the products

$$f_i f_j^{-1},$$

for $0 \le i < j \le r$, and show that $\mathscr{V}$ is a refinement of $\mathscr{U}$. Consider $V \in \mathscr{V}$. On the set $S = \{0, \dots, r\}$ we define a relation $\ll$ by $i \ll j$ if $|f_i(x)| < |f_j(x)|$ for all $x \in V$. One readily verifies that for all distinct $i, j \in S$, we either have $i \ll j$ or $j \ll i$. Thus there is a maximal element $i_S \in S$ with respect to $\ll$, and then

$$V \subseteq X\left(\frac{f_0, \dots, f_r}{f_{i_S}}\right).$$

Therefore $\mathscr{V}$ is a refinement of $\mathscr{U}$. □

The upshot of the previous five lemmas[2] is stated in the following:

**Proposition 4.10.** *Let $\mathscr{F}$ be a presheaf on an affinoid space $X$. If $\mathscr{F}$ is a $\mathscr{U}$-sheaf for all Laurent coverings $\mathscr{U}$ of $X$, then it is a $\mathscr{V}$-sheaf for all affinoid covers $\mathscr{V}$.*

Thus, the task of proving the theorem for all affinoid covers is reduced to proving it for Laurent covers. What's more is that an inductive argument yields it is only necessary to consider Laurent coverings generated by a single element $f \in \mathcal{O}_X(X)$. Recall that by $A\langle f\rangle$ we understand the algebra $A\langle t\rangle/(t - f)$, which is the algebra representing the simple Laurent domain $X(f)$. Then our goal is to show the sequence

$$0 \to A \xrightarrow{\varepsilon} A\langle f\rangle \times A\langle f^{-1}\rangle \xrightarrow{\delta} A\langle f, f^{-1}\rangle \qquad (**)$$

is exact. The map $\varepsilon$ is defined

$$\varepsilon\colon f \mapsto (f|_{X(f)}, f|_{X(f^{-1})}),$$

and the map $\delta$ is given as

$$\delta\colon (f, g) \mapsto f|_{X(f,f^{-1})} - g|_{X(f,f^{-1})}.$$

Now the sequence $(**)$ fits into the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\zeta - f)A\langle\zeta\rangle \times (1 - f\eta)A\langle\eta\rangle & \xrightarrow{\delta''} & (\zeta - f)A\langle\zeta, \zeta^{-1}\rangle & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varepsilon'} & A\langle\zeta\rangle \times A\langle\eta\rangle & \xrightarrow{\delta'} & A\langle\zeta, \zeta^{-1}\rangle & \longrightarrow & 0 \\
 & & \text{id}\updownarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varepsilon} & A\langle f\rangle \times A\langle f^{-1}\rangle & \xrightarrow{\delta} & A\langle f, f^{-1}\rangle & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

in which $\zeta, \eta$ are indeterminates. The map $\varepsilon'$ is the natural inclusion and $\delta'$ is given by $(h_1(\zeta), h_2(\eta)) \mapsto h_1(\zeta) - h_2(\zeta^{-1})$. Then $\delta''$ is induced by $\delta'$. Finally the vertical maps are given by $\zeta \mapsto f$ and $\eta \mapsto f^{-1}$. The first column is exact from the definition of $A\langle f\rangle$ and $A\langle f^{-1}\rangle$. The second column is exact because

$$A\langle f, f^{-1}\rangle = A\langle\zeta, \eta\rangle/(\zeta - f, 1 - f\eta) = A\langle\zeta, \zeta^{-1}\rangle/(\zeta - f).$$

The surjectivity of $\delta'$ is clear. The same holds for $\delta''$, as

$$(\zeta - f)A\langle\zeta, \zeta^{-1}\rangle = (\zeta - f)A\langle\zeta\rangle + (1 - f\zeta^{-1})A\langle\zeta^{-1}\rangle.$$

Thus the first row is exact. The second row is also readily verified to be exact. The map $\varepsilon$ is injective by ([Bos14], Ch. 4 §1 cor. 5). Then, exactness of the third row (as we require) is a diagram

[2] not to be confused with the five lemma :)

chase. □

With this we have shown that, with respect to affinoid covers, the presheaf $\mathcal{O}_X$ satisfies the sheaf condition.

## 5 Rigid Spaces and Analytification

The content of the previous section amounts to saying we have a way to define a sheaf of an affinoid space $X$, given that we only allow a certain class of coverings of our space. The machinery for dealing with a situation like this is that of a *Grothendieck topological space.* It is a sort of generalization of a topological space.

**Definition 5.1.** A *Grothendieck topological space* (often shortened to $G$-topological space) is a triple $(X, \mathscr{C}, \mathrm{Cov}(X))$, where $X$ is a set, $\mathscr{C}$ is full subcategory of the subsets of $X$ with inclusions as morphisms, and $\mathrm{Cov}(X)$ is family of coverings $\mathscr{U} = (U_i)_{i\in I}$ of subsets $U \in \mathscr{C}$ by $U_i \in \mathscr{C}$ satisfying the following axioms:

1. For any $U \in \mathscr{C}$, the trivial covering $\mathscr{U} = \{U\}$ is in $\mathrm{Cov}(X)$.
2. If $\mathscr{U} = (U_i)_{i\in I} \in \mathrm{Cov}(X)$, and for each $i$ there a covering $\mathscr{V}_i = (V_{i,j})_{j\in J_i}$ of $U_i$ in $\mathrm{Cov}(X)$, then $\mathscr{V} = (V_{i,j})_{i,j} \in \mathrm{Cov}(X)$.
3. If $(U_i)_{i\in I} \in \mathrm{Cov}(X)$ and $V \in \mathscr{C}$, then the covering $(U_i \cap V)_{i\in I}$ of $V$ is in $\mathrm{Cov}(X)$.

With this we define a more general notion of a sheaf.

**Definition 5.2.** A *sheaf* $\mathscr{F}$ of objects in a Abelian category $\mathcal{A}$ on a $G$-topological space $(X, \mathscr{C}, \mathrm{Cov}(X))$ is a contravariant functor

$$\mathscr{F} : \mathscr{C} \to \mathcal{A}$$

such that for any covering $\mathscr{U} \in \mathrm{Cov}(X)$ of $U \in \mathscr{C}$, the sequence

$$0 \to \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \to \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is exact.

The *weak Grothendieck topology* on an affinoid space $X$ is the triple $(X, \mathscr{T}, \mathrm{Cov}(X))$ where $\mathscr{T}$ is category of affinoid subdomains and $\mathrm{Cov}(X)$ is the family of affinoid coverings. One checks that this satisfies the axioms of being a $G$-topological space. Now section 4 is recapitulated in this way: The presheaf $\mathcal{O}_X$ is a sheaf on the weak Grothendieck topology $(X, \mathscr{T}, \mathrm{Cov}(X))$. We now extend the weak Grothendieck topology to allow more covers.

**Definition 5.3.** Let $X$ be an affinoid space. The *strong Grothendieck topology* on $X$ is given as follows:

1. A subset $U \subseteq X$ is called an *admissible open* if there is a covering $U = \bigcup_{i\in I} U_i$ of $U$ by affinoid subdomains $U_i \subseteq X$ such that for all maps of affinoid spaces $\varphi\colon Z \to X$ with $\varphi(Z) \subseteq U$, the covering $(\varphi^{-1}(U_i))_{i\in I}$ of $Z$ admits a refinement that is a finite covering of $Z$ by affinoid subdomains.
2. A covering $V = \bigcup_{j\in J} V_j$ of some admissible open $V \subseteq X$ by admissible open sets $V_j$ is an *admissible cover* if for each map of affinoid spaces $\varphi\colon Z \to X$ with $\varphi(Z) \subseteq V$, the covering $(\varphi^{-1}(V_j))_{j\in J}$ of $Z$ admits a refinement that is a finite covering of $Z$ by affinoid subdomains.

One also verifies that the strong Grothendieck topology on $X$, $(X, \mathscr{I}, \mathrm{Cov}(X))$, where $\mathscr{I}$ is the admissible opens and $\mathrm{Cov}(X)$ is the admissible covers, really is a $G$-topological space. One benefit of the strong Grothendieck topology is that it satisfies so-called "completeness conditions" which will allow the construction of Grothendieck topologies on global spaces from local spaces.

**Proposition 5.4.** *Let $X$ be an affinoid space. Then the strong Grothendieck topology on $X$ satisifes the following completeness conditions:*

1. *$\emptyset$ and $X$ are admissible opens.*
2. *Let $(U_i)_{i\in I}$ be an admissible covering of an admissible open $U \subseteq X$. Let $V \subseteq X$ be a subset such that $V \cap U_i$ is an admissible open for all $i \in I$. Then $V$ is an admissible open.*
3. *Let $(U_i)_{i\in I}$ be a covering of an admissible open $U \subseteq X$ by admissible opens $U_i$ such that $(U_i)_{i\in I}$ admits an admissible covering of $U$ as a refinement. Then $(U_i)_{i\in I}$ is admissible.*

We now are able to extend the presheaf $\mathcal{O}_X$ (which is a sheaf in the weak Grothendieck topology) to a sheaf on the strong Grothendieck topology. Let $U \subseteq X$ be any admissible open and $\mathscr{U} = (U_i)_{i\in I}$ an admissible covering of $U$ by affinoids. We then put

$$\mathcal{O}_X(U) = \ker\left(\prod_i \mathcal{O}_X(U_i) \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)\right).$$

**Proposition 5.5.** *The presheaf $\mathcal{O}_X$ as defined above on an affinoid space with the strong Grothendieck topology $(X, \mathscr{I}, \mathrm{Cov}(X))$ is a sheaf.*

*Proof.* See ([Bos14], Ch. 5 §2, cor. 5). ☐

By a *$G$-ringed space* we mean a $G$-topological space together with a structure sheaf of rings. These are nice because stalks are easy to define:

**Definition 5.6.** Let $(X, \mathcal{O}_X)$ be a $G$-ringed space. The for any $x \in X$, the *stalk* at $x$ is

$$\mathcal{O}_{X,x} = \varinjlim_{x\in U\subseteq X} \mathcal{O}_X(U),$$

$U$ ranging over the admissible opens in which $x$ is an element.

**Lemma 5.7.** *Let $X$ be an affinoid space equipped with the strong Grothendieck topology. Then, for $x \in X$, $\mathcal{O}_{X,x}$ is a local ring.*

*Proof.* See ([Bos14], Ch. 4 §1, prop. 1). ☐

**Definition 5.8.** A *locally $G$-ringed $K$-space* is a pair $(X, \mathcal{O}_X)$ where $X$ is a $G$-topological space and $\mathcal{O}_X$ is a sheaf of $K$-algebras such that all the stalks $\mathcal{O}_{X,x}$ are local rings.

A *morphism* of locally $G$-ringed $K$-spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair $(\varphi, \varphi^*)$ where $\varphi\colon X \to Y$ is a morphism of $G$-topological spaces and $\varphi^*\colon \mathcal{O}_Y \to \mathcal{O}_X$ is a morphism of sheaves of $K$-algebras, such that, for each $x \in X$, the induced morphism

$$\mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$$

is local.

**Proposition 5.9.** *Any affinoid space $X$ equipped with the strong Grothendieck topology is a locally $G$-ringed $K$-space $(X, \mathcal{O}_X)$. In other words, the association $A \rightsquigarrow (X = \mathrm{Spec_m}\, A, \mathcal{O}_X)$ defines a fully faithful contravariant functor*

$$\mathrm{AffAlg}_K \to \{\textit{Locally } G\textit{-ringed } K\textit{-spaces}\}.$$

We say that a locally $G$-ringed $K$-space is an *affinoid rigid space* if it is in the essential image of the functor defined in proposition 5.9. With this we are finally able to define rigid analytic spaces, in the sense of Tate.

**Definition 5.10.** A *rigid analytic space* is a locally $G$-ringed $K$-space $(X, \mathcal{O}_X)$ satisfying conditions

1. If $(U_i)_{i \in I}$ is an admissible cover of an admissible $U \subseteq X$ and $V \subseteq X$ is a subset such that $V \cap U_i$ is an admissible open for each $i$, then $V$ is an admissible open.

2. Any covering $\mathscr{U}$ of an admissible open $U$ that can be refined by an admissible covering is admissible.

and such that $X$ admits an admissible cover $X = \bigcup_{i \in I} U_i$ where each $(U_i, \mathcal{O}_X|_{U_i})$ is an affinoid rigid space.

Finally, we state some theorems relating to analytification:

**Theorem 5.11.** *Let $X$ be any scheme of locally finite type over $K$. Then there is a rigid space $X^{\text{an}}$ and a morphism of locally $G$-ringed $K$-spaces*

$$X^{\text{an}} \to X$$

*such that for any rigid space $Y$ and morphism $Y \to X$ of locally $G$-ringed $K$-spaces, there is a unique morphism $Y \to X^{\text{an}}$ making*

$$\begin{array}{ccc} X^{\text{an}} & \longrightarrow & X \\ & \nwarrow & \uparrow \\ & & Y \end{array}$$

*commute. Furthermore, the map $X^{\text{an}} \to X$ induces a bijection between $X^{\text{an}}$ and the closed points of $X$.*

*Proof.* See ([HS22], prop. 162). □

**Theorem 5.12** (Rigid GAGA)**.** *Let $X, Y$ be proper $K$-schemes. Then* $\text{Mor}(X, Y) \to \text{Mor}(X^{\text{an}}, Y^{\text{an}})$ *is bijective. In particular, analytification identifies categories of coherent modules on $X$ and $X^{\text{an}}$.*

A Zariski closed subspace $Z \subseteq \mathbb{P}^{n,\text{an}}$ is called a *projective rigid space.*

**Corollary 5.13** (Chow's Theorem)**.** *Any projective rigid space is the analytification of a projective $K$-scheme.*

This allows one to go back and forth between projective schemes over $K$ and rigid spaces. Thus one can utilize analytic methods in the rigid spaces world and transport that back to the scheme setting.

## References


[BGR12] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry.* Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012.

[Bos14] S. Bosch. *Lectures on Formal and Rigid Geometry.* Lecture Notes in Mathematics. Springer International Publishing, 2014.

[Con08] B. Conrad. Several approaches to non-archimedean geometry. *p-adic geometry*, pages 9–63, 2008.

[GG69] Von Lothar Gerritzen and Hans Grauert. *Die Azyklizität der affinoiden Überdeckungen*, pages 159–184. Princeton University Press, 1969.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977.

[HS22] B. Heuer and M. Stevens. Rigid analytic geometry, 2022.

[Sch11] P. Schneider. *p-Adic Lie Groups*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2011.

[SG13] J.P. Serre and M.J. Greenberg. *Local Fields*. Graduate Texts in Mathematics. Springer New York, 2013.

[Tat71] J. Tate. Rigid analytic spaces. *Inventiones mathematicae*, 12:257–289, 1971.

[Tem05] M. Temkin. A new proof of the gerritzen-grauert theorem. *Mathematische Annalen*, 333(2):261–269, Oct 2005.

[Tia] Y. Tian. Introduction to rigid geometry.